# Subject CM1

## CMP Upgrade 2023/24

### CMP Upgrade

This CMP Upgrade lists the changes to the Syllabus, Core Reading and the ActEd material since last year that might realistically affect your chance of success in the exam. It is produced so that you can manually amend your 2023 CMP to make it suitable for study for the 2024 exams. It includes replacement pages and additional pages where appropriate.

Alternatively, you can buy a full set of up-to-date Course Notes / CMP at a significantly reduced price if you have previously bought the full-price Course Notes / CMP in this subject. Please see our 2024 *Student Brochure* for more details.

We only accept the current version of assignments for marking, *ie* those published for the sessions leading to the 2024 exams. If you wish to submit your scripts for marking but only have an old version, then you can order the current assignments free of charge if you have purchased the same assignments in the same subject in a previous year, and have purchased marking for the 2024 session.

This CMP Upgrade contains:

- all significant changes to the Syllabus and Core Reading
- additional changes to the ActEd Course Notes and Assignments that will make them suitable for study for the 2024 exams.

 

# 0 Retaker discounts

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Students have the choice of purchasing the full CMP (printed or eBook) or just the Course Notes (printed).

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# 1 Changes to the Syllabus

There have been a large number of changes to the wordings of the syllabus objectives and topic weightings have changed. The updated syllabus objectives are set out in full below.

***Objectives***

**1. Theory of interest rates (25%)**

Understand the principles of time preference theory of interest and the time value of money, including the term structure of interest rates and standard actuarial compound interest rate functions. Apply these principles to real world examples of interest rates, discounting and evaluation of present value of cashflows.

1.1 Show how interest rates may be expressed in different time periods: (Chapters 1 and 2)

1.1.1 Relationship between the rates of interest and discount over one effective period, considered arithmetically and by general reasoning.

1.1.2 Determine, when given a rate of interest under a specific payment frequency, the equivalent rate under an alternative payment frequency including:

- annual effective rate of interest or discount
- rate of interest of discount payable pthly (p>1)
- force of interest.

1.1.3 Calculate the equivalent annual rate of interest implied by the accumulation of a sum of money over a specified period where the force of interest is a function of time.

1.2 Account for the time value of money using the concepts of compound interest and discounting: (Chapter 1)

1.2.1 Accumulate a single investment at a constant rate of interest under the operation of simple and compound interest.

1.2.2 Calculate the present value of a future payment by discounting a single investment.

1.3 Extend the techniques in 1.1 and 1.2 where appropriate to allow for inflation. (Chapters 3 and 10)

1.4 Describe the operation of financial instruments and insurance contracts as a cashflow model (where cashflows may be fixed or uncertain in terms of both amount and timing). (Chapters 7, 8 and 10)

1.5 Calculate the present value and accumulated value for a given stream of cashflows under the following individual combination of scenarios: (Chapter 4)

1.5.1 Cashflows are equal at each time period;

1.5.2 Cashflows vary with time, which may or may not be a continuous function of time;

1.5.3 Some of the cashflows are deferred for a long period of time;

1.5.4 Rate of interest or discount is constant; and

1.5.5 Rate of interest or discount varies with time, which may or may not be a continuous function of time.

1.6 Evaluate the following annuity and accumulation functions, when given the values for the term, $n$, and the appropriate interest or discount function $i$, $v$, $d$, $\delta$, $i^{(p)}$ and $d^{(p)}$: (Chapters 5 and 6)

1.6.1 $a_{\overline{n}|}$, $s_{\overline{n}|}$, $a^{(p)}_{\overline{n}|}$, $s^{(p)}_{\overline{n}|}$, $\ddot{a}_{\overline{n}|}$, $\ddot{s}_{\overline{n}|}$, $\ddot{a}^{(p)}_{\overline{n}|}$, $\ddot{s}^{(p)}_{\overline{n}|}$, $\bar{a}_{\overline{n}|}$ and $\bar{s}_{\overline{n}|}$.

1.6.2 ${}_{m|}a_{\overline{n}|}$, ${}_{m|}a^{(p)}_{\overline{n}|}$, ${}_{m|}\ddot{a}_{\overline{n}|}$, ${}_{m|}\ddot{a}^{(p)}_{\overline{n}|}$ and ${}_{m|}\bar{a}_{\overline{n}|}$.

1.6.3 $(Ia)_{\overline{n}|}$, $(I\ddot{a})_{\overline{n}|}$, $(I\bar{a})_{\overline{n}|}$ and $(\bar{I}\bar{a})_{\overline{n}|}$ and the respective deferred annuities.

1.7 Demonstrate an understanding of the term structure of interest rates: (Chapter 11)

1.7.1 Understand the main factors influencing the term structure of interest rates.

1.7.2 Understand and calculate:

- discrete spot rates and forward rates.
- continuous spot rates and forward rates.

1.7.3 Understand and calculate the par yield and yield to maturity.

1.8 Understand duration, convexity and immunisation of cashflows: (Chapter 11)

1.8.1 Demonstrate how the duration and convexity of a cashflow sequence may be used to estimate the sensitivity of the value of the cashflow sequence to a shift in interest rates.

1.8.2 Understand, apply and discuss Redington's conditions for immunisation of a portfolio of liabilities.

**2. Equation of value and its applications (20%)**

Understand and apply equation of value principles to evaluate financial problems, in particular relating to loan schedules, bond prices, bond yields and project appraisals:

2.1 Understand an apply the concept of an equation of value in terms of: (Chapter 7)

- where payment or receipt is certain
- where payment or receipt is uncertain
- the two conditions required for there to be an exact solution

2.2 Use the concept of equation of value to solve various practical problems: (Chapters 8 and 10)

2.2.1 Apply the equation of value to loans repaid by regular instalments of interest and capital. Obtain repayments, interest and capital components, the effective interest rate (APR) and construct a schedule of repayments.

2.2.2 Calculate the price of, or yield (nominal or real allowing for inflation) from, a bond (fixed-interest or index-linked) where the investor is subject to deduction of income tax on coupon payments and redemption payments are subject to deduction of capital gains tax.

2.2.3 Calculate the running yield and the redemption yield for the financial instrument as described in 2.2.2.

2.2.4 Calculate the upper and lower bounds for the present value of the financial instrument as described in 2.2.2, when the redemption date can be a single date within a given range at the option of the borrower.

2.2.5 Calculate the present value or yield (nominal or real allowing for inflation) from an ordinary share or property, given constant or variable rate of growth of dividends or rents.

2.3 Apply cashflow and equation of value techniques to project appraisals: (Chapter 9)

2.3.1 Calculate the net present value and accumulated profit of the receipts and payments from an investment project at given rates of interest.

2.3.2 Calculate the internal rate of return, payback period and discounted payback period and discuss their suitability for assessing the suitability of an investment project.

## 3. Decrement and multiple life models (20%)

Understand how to model uncertain future cashflows, which may depend on the death or survival of an individual, or other uncertain events. Be introduced to the life table, calculation of the mean and variance of the present value of all of the main life insurance and annuity contracts, and their relationship in actuarial terms. Extend the single decrement model to evaluate health insurance contracts involving two lives as well as the valuation of cashflows in a competing risk environment using multiple state models:

3.1 Demonstrate an understanding of the operation of key assurance and annuity contracts: (Chapters 12, 13, 14, 16 and 23)

3.1.1 Understand the following contracts, for example by explaining the timing and nature of the cashflows involved:

- whole-life assurance
- term assurance
- pure endowment
- endowment assurance
- whole-life level annuity
- temporary level annuity
- guaranteed level annuity
- deferred benefits

3.1.2 Understand the operation of conventional with-profits contracts, where profits are distributed by the use of regular reversionary bonuses and by terminal bonuses.

3.1.3 Understand the operation of conventional unit-linked contracts, where death benefits are expressed as combination of absolute amount and relative to a unit fund.

3.1.4 Understand the operation of accumulating with-profits contracts where benefits take the form of an accumulating fund of premiums, where:

- the fund is defined in monetary terms, has no explicit charges, and is increased by the addition of regular guaranteed and bonus interest payments plus a terminal bonus; or
- the fund is defined in terms of the value of a unit fund, is subject to explicit charges, and is increased by regular bonus additions (through unit price increases or allocations of additional units) plus a terminal bonus (unitised with-profits).

3.2 Apply formulae for the means and variances of the payments under various assurance and annuity contracts, assuming constant deterministic interest rate:
(Chapters 12, 13, 14, 15, 16 and 18)

3.2.1 Life table functions $l_x$ and $d_x$ and their select equivalents $l_{[x]+r}$ and $d_{[x]+r}$.

3.2.2 Describe the meaning of the following probabilities: ${}_np_x$, ${}_nq_x$, ${}_{n|m}q_x$, ${}_{n|}q_x$ and their select equivalents ${}_np_{[x]+r}$, ${}_nq_{[x]+r}$, ${}_{n|m}q_{[x]+r}$, ${}_{n|}q_{[x]+r}$.

3.2.3 Express the probabilities defined in 3.2.2 in terms of life table functions defined in 3.2.1.

3.2.4 Use assurance and annuity factors and their select and continuous equivalents including the extension of the annuity factors to allow for the possibility that payments are more frequent than annual but less frequent than continuous.

3.2.5 Use the relationship between annuities payable in advance and in arrear, and between temporary, deferred and whole life annuities.

3.2.6 Use the relationship between assurance and annuity factors using equation of value, and their select and continuous equivalents.

3.2.7 Obtain the mean and variance of the present value of benefit payments as sums / integrals under each contract defined in 3.1.1, in terms of the (curtate) random future lifetime, assuming:

- contingent benefits (constant, increasing or decreasing) are payable at the middle or end of the year of contingent event or continuously.
- annuities are paid in advance, in arrear or continuously, and the amount is constant, increases or decreases by a constant monetary amount or by a fixed or time-dependent variable rate.
- premiums are payable in advance, in arrear or continuously; and for the full policy term or for limited period.

Where appropriate, simplify the above expressions into a form suitable for evaluation by table look-up or other means.

3.2.8 Evaluate the expected accumulations in terms of expected values for the contracts described in 3.1.1 and contract structures described in 3.2.7.

3.3 Describe and use assurance and annuity functions involving two lives: (Chapters 19 and 20)

3.3.1 Extend the techniques of objectives 3.2 to deal with cashflows dependent upon the death or survival of either or both of two lives.

3.3.2 Extend technique in 3.3.1 to deal with functions dependent upon a fixed term as well as age.

3.4 Describe and apply methods of valuing cashflows that are contingent upon multiple transition events: (Chapter 22)

3.4.1 Demonstrate an understanding of simple health insurance premium and benefit structures.

3.4.2 Describe how a cashflow, contingent upon multiply transition events, may be valued using a multiple state Markov model, in terms of the forces and probabilities of transition.

3.4.3 Construct formulae for the expected present values of cashflows that are contingent upon multiple transition events, including simple health insurance premiums and benefits, and calculate these in simple cases. This includes regular premiums and sickness benefits that are payable continuously and assurance benefits that are payable immediately on transition.

3.5 Describe and use methods of projecting and valuing expected cashflow that are contingent upon multiple decrement events: (Chapter 22)

3.5.1 Understand the construction and use of multiple decrement tables.

3.5.2 Understand the operation of a multiple decrement model as a special case of multiple-state Markov model.

3.5.3 Determine dependent probabilities for a multiple decrement model in terms of given forces of transition, assuming forces of transition are constant over single years of age.

3.5.4 Determine forces of transition from given dependent probabilities, assuming forces of transition are constant over single years of age.

## 4. Pricing and reserving (35%)

Understand the future loss random variable and its application to the calculation of premiums for conventional life assurance and annuity contracts. Use the prospective and retrospective approaches to calculate reserves, the recursive relationship between reserves, and calculate mortality profit. Project cashflows to profit test life insurance contracts and apply projected cashflow techniques to pricing and reserving:

4.1 Determine the gross random future loss random variable under an insurance contract. (Chapter 18)

4.2 Calculate gross premiums and reserves of assurance and annuity contracts: (Chapters 17 and 18)

4.2.1 Calculate gross premiums for the insurance contract benefits listed in 3.1.1 under the following scenarios, or a combination thereof, using the equivalence principle or otherwise:

- contracts may accept only a single premium
- regular premiums and annuity benefits may be payable annually, more frequently than annually or continuously
- death benefits (which increase or decrease by a constant compound rate or by a constant monetary amount) may be payable at the end of the year of death or immediately on death
- survival benefits (other than annuities) may be payable at defined intervals other than at maturity.

4.2.2 Understand why an insurance company will set up reserves.

4.2.3 Calculate gross prospective and retrospective reserves.

4.2.4 Understand the equivalence of the prospective reserve and the retrospective reserve under certain conditions, with or without allowance for expenses, for all fixed benefit and increasing/decreasing benefit contracts.

4.2.5 Obtain recursive relationships between successive periodic gross premium reserves, and use this relationship to calculate the profit earned from a contract during the period.

4.2.6 Understand the concepts of net premiums and net premium valuation and how they relate to gross premiums and gross premium valuation respectively.

4.3 Describe and calculate, for a single policy or a portfolio of policies (as appropriate):

- death strain at risk;
- expected death strain;
- actual death strain; and
- mortality profit

for policies with death benefits payable immediately on death or at the end of the year of death, policies paying annuity benefits at the start of the year or on survival to the end of the year, and policies where single or non-single premiums are payable. (Chapter 21)

4.4 Project expected future cashflows for whole life, endowment and term assurances, annuities, unit-linked contracts, and conventional/unitised with-profits contracts, incorporating multiple decrement models as appropriate: (Chapters 24 and 25)

4.4.1 Profit test life insurance contracts of the types listed above and determine the profit vector, the profit signature, the net present value, and the profit margin.

4.4.2 Show how a profit test may be used to price a product, and use a profit test to calculate a premium for life insurance contracts of the types listed above.

4.4.3 Show how gross premium reserves can be computed using the above cashflow projection model and included as part of profit testing.

4.5 Show how, for unit-linked contracts, non-unit reserves can be established to eliminate ('zeroise') future negative cashflows, using a profit test model. (Chapter 25)

# 2 Changes to the Core Reading

This section contains all the ***non-trivial*** changes to the Core Reading.

## Chapters 1 & 2

The Core Reading from Chapters 1 & 2 has largely been removed. The retained Core Reading has been moved to Chapter 9 (see below).

## Chapter 9 (now Chapter 7)

Core Reading relating to fixed-interest securities, index-linked securities and cash on deposit have been relocated to Chapter 9 from Chapter 2. Core Reading on stochastic and deterministic models and sensitivity testing have been relocated to Chapter 9 from Chapter 1. The new Chapter 9 is appended in full to this upgrade.

## Chapter 19 (now Chapter 17)

**Section 3.3**

A new paragraph of Core Reading has been added at the very end of Section 3.3:

**The formulae in this and the following chapter all require the independence of decrements assumptions, as defined in syllabus objective 4.4.2 to hold. Therefore, the application of lapses and surrender is largely ignored, however could be examined as part of the computer based assessment where cashflow models could be used instead.**

**Section 3.4**

A line has been added to the second bullet point at the opening to Section 3.4.

**Note: A variant of the annual premium contract is limited premium paying term contracts, where the premium term is less than the contract term.**

## Chapter 20 (now Chapter 18)

**Section 0**

The following paragraph of Core Reading has been added to the introduction:

**It is important to note that policyholder behaviour is a key consideration for insurers in reserving and profit testing of life insurance business. For instance, in insurance lines where contracts are regularly renewed (usually annually), insurers are mainly concerned with the extent to which policyholders renew or continue their contracts. As a result, lapses and surrenders are major decrements that impact pricing and reserving in practice. Additionally the mortality experience of life insurance companies may depend on their lapses and surrender experience.**

## Chapter 24 (now Chapter 22)

### Section 3.3

The opening paragraph to Section 3.3 leads into the wrong equation. It should read:

**We can use the Kolmogorov forward differential equations to derive transition probabilities, as in the case of multiple state models. We note from Section 2.1 that, in the multiple state model, this produces the following general result:**

$$ {}_t p_x^{\overline{ii}} = \exp\left( -\int_0^t \sum_{j \neq i} \mu_{x+s}^{ij} \, ds \right) $$

### Section 4

The form of the deferred dependent probability on page 26 is incorrect. It has been corrected to read:

**We can also use the table to calculate *deferred* dependent probabilities of the form:**

$$ {}_{n|}(aq)_x^k = \frac{(ad)_{x+n}^k}{(al)_x} $$

A line of Core Reading on page 27 contains the wrong notation. The corrected notation is below.

**The notation used is $l_x^j, d_x^j, q_x^j, p_x^j, \mu_x^j$ *etc* for mode of decrement $j$.**

# 3 Changes to the ActEd material

This section contains all the ***non-trivial*** changes to the ActEd text.

## Chapters 1 & 2

These chapters have been removed. All subsequent chapter numbers and references have been reduced by 2. The chapter numbers below refer to the 2023 version of the Course Notes.

## Chapter 9 (now Chapter 7)

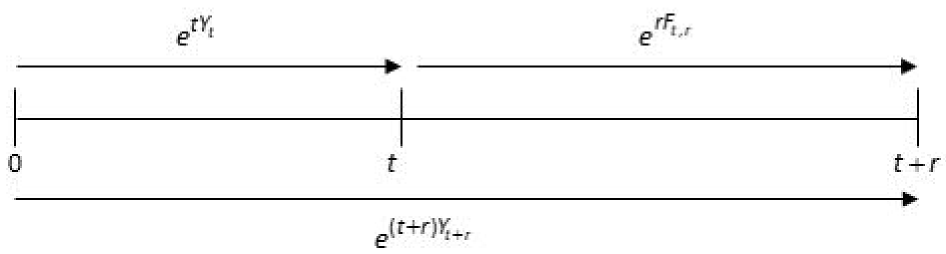
There have been a number of changes to this chapter. The new Chapter 9 is appended in full to this upgrade.

## Chapter 13 (now Chapter 11)

**Page 10**

A formatting error had interfered with the timeline diagram at the top of the page. It has been corrected to read as follows:

Once again, we can represent the connection between the continuous-time spot and forward rates on a timeline.




## Chapter 17 (now Chapter 15)

**Page 16**

The first non-bold equation on page 17 contains an erroneous n subscript on the left-hand side. The equation has been corrected to read:

$$\ddot{a}_x^{(m)} = \frac{1}{m} + a_x^{(m)}$$

## Chapter 20 (now Chapter 18)

**Page 17**

The solution to the question at the bottom of page 16 was incorrect. It has been corrected to the following:

The accumulated fund will be:

$$1{,}000{,}000 \times 1.04^{20} = 2{,}191{,}123$$

The expected number of survivors will be:

$$10{,}000 \times \frac{l_{60}}{l_{40}} = 10{,}000 \times \frac{9{,}287.2164}{9{,}856.2863} = 9{,}422.63$$

So the expected payout per survivor is:

$$\frac{2{,}191{,}123}{9{,}422.63} = £232.54$$

## Chapter 26 (now Chapter 24)

**Page 24**

The table at the top of the page was incorrect as the expected claim expense had not been allowed for in the expected profit per policy in force at the start of the year.

The table now reads:

| Year | Premium | Expense | Interest | Expected claim expense | Expected death cost | Expected maturity cost | Expected profit per policy in force at start of year |
|---|---|---|---|---|---|---|---|
| 1 | 2,000 | –200 | 90.00 | –5.00 | -104.00 | | 1,781.00 |
| 2 | 2,000 | –30 | 98.50 | –5.10 | -216.40 | | 1,847.00 |
| 3 | 2,000 | –30 | 98.50 | –5.20 | -337.63 | | 1,725.67 |
| 4 | 2,000 | –30 | 98.50 | –5.30 | -468.13 | | 1,595.07 |
| 5 | 2,000 | –30 | 98.50 | –100 | -608.36 | -11,723.35 | -10,363.21 |

In the solution section there was also an error in the interest earned in year 5, which has been corrected to:

$$(2{,}000 - 30) \times 0.05 = 98.50$$

Hence the verification of the expected profit per policy in force at start of year 5 should be:

$$2,000-30+98.50-100-608.36-11,723.35=-10,363.21$$

# 4 Changes to the X Assignments

## Overall

There have been minor changes throughout the assignments, including changes to mark allocations.

More significant changes are listed below.

## Assignment X1

A number of changes have been made to the questions and solutions. Replacement pages can be found at the end of this Upgrade.

## Assignment X2

***X2.2***

There has been a minor rewording to this question:

A borrower has agreed to repay a loan of £20,000 with payments of £427.90 made monthly in arrears for 5 years.

Calculate, without using Goal Seek or Solver and showing all workings, the APR charged on the loan.

***X2.7***

There has been a minor rewording to part (i):

(i) Show that the amount of the loan is approximately £38,000. [3]

## Assignment X3

***X3.5***

Part (ii) has been removed and adjustments made to the marks in parts (i) and (iii):

A man aged exactly 42 purchases a whole life assurance policy with a sum assured of 5,000 payable immediately on death.

(i) Write down an expression for the random variable representing the present value of the benefits from this policy. [1]

(ii) Calculate the variance of the present value of the benefits from this assurance policy, assuming AM92 Ultimate mortality and 4% *pa* interest. [3]

[Total 4]

***X3.12***

Part (ii) has been removed and marks updated in parts (i) and (iii):

Let $K$ denote the curtate future lifetime random variable of a life aged exactly $x$.

(i) Describe the benefit whose present value random variable is:

$$W = \begin{cases} 10{,}000\ddot{a}_{\overline{K+1}|} & \text{if } K < 10 \\ 10{,}000\ddot{a}_{\overline{10}|} & \text{if } K \geq 10 \end{cases}$$ [2]

(ii) Calculate the expected present value and the standard deviation of the present value of the benefit in (i), assuming:

- a force of interest of 0.04 *pa*
- the life is subject to a constant force of mortality of 0.02 *pa*. [9]

[Total 11]

## Assignment X4

***X4.5***

This question has been removed.

***X4.9***

Part (iii) has been removed.

## Assignment X5

***X5.4***

This question has been removed.

***X5.6***

The alternative solution presented for part (ii) contained a couple of errors. The term and following explanation has been corrected to read:

$$50{,}000\int_1^{10} e^{-\delta t}\ {}_{t-1}p_{50}^{ai}\ {}_1p_{50+t-1}^{\overline{ii}}\,\upsilon_{50+t}\ dt$$

*This time the PDF is* ${}_{t-1}p_{50}^{ai}\ {}_1p_{50+t-1}^{\overline{ii}}$, *that is at time t the life was sick one year ago, has remained sick for the last year and then has died from the sick state. This can be evaluated between the limits of 1 and 10.*

# 5 Changes to the Y Assignments

There are no changes to the Y assignments.

# 6 Other tuition services

In addition to the CMP you might find the following services helpful with your study.

## 6.1 Study material

We also offer the following study material in Subject CM1:

- Flashcards
- Revision Notes
- ASET (ActEd Solutions with Exam Technique) and Mini-ASET
- Mock Exam and AMP (Additional Mock Pack).

For further details on ActEd's study materials, please refer to the 2024 *Student Brochure*, which is available from the ActEd website at **ActEd.co.uk**.

## 6.2 Tutorials

We offer the following (face-to-face and/or online) tutorials in Subject CM1:

- a set of Regular Tutorials (lasting a total of five days)
- a Split Block Tutorial (lasting five full days)
- a Preparation Day for the Paper B exam
- six-day Bundles in both Regular and Block format, covering the five days for the Paper A exam, plus the Preparation Day for the Paper B exam
- an Online Classroom.

For further details on ActEd's tutorials, please refer to our latest *Tuition Bulletin*, which is available from the ActEd website at **ActEd.co.uk**.

## 6.3 Marking

You can have your attempts at any of our assignments or mock exams marked by ActEd. When marking your scripts, we aim to provide specific advice to improve your chances of success in the exam and to return your scripts as quickly as possible.

For further details on ActEd's marking services, please refer to the 2024 *Student Brochure*, which is available from the ActEd website at **ActEd.co.uk**.

## 6.4 Feedback on the study material

ActEd is always pleased to receive feedback from students about any aspect of our study programmes. Please let us know if you have any specific comments (*eg* about certain sections of the notes or particular questions) or general suggestions about how we can improve the study material. We will incorporate as many of your suggestions as we can when we update the course material each year.

If you have any comments on this course, please send them by email to **CM1@bpp.com**.

# 7

# Equations of value

## Syllabus objectives

1.4 Describe the operation of financial instruments and insurance contracts as a cashflow model (where cashflows may be fixed or uncertain in terms of both amount and timing).

2.1 Understand and apply the concept of an equation of value in terms of:

- where payment or receipt is certain.
- where payment or receipt is uncertain
- the two conditions required for there to be an exact solution.

# 0 Introduction

An *equation of value* equates the present value of money received to the present value of money paid out:

*'PV income = PV outgo'*

or equivalently:

'*PV income – PV outgo = 0'*

Equations of value are used throughout actuarial work. For example:

- the 'fair price' to pay for an investment such as a fixed-interest security or an equity (*ie PV outgo*) equals the present value of the proceeds from the investment, discounted at the rate of interest required by the investor.
- the premium for an insurance policy is calculated by equating the present value of the expected amounts received in premiums to the present value of the expected benefits and other outgo.

# 1 The equation of value and the yield on a transaction

## 1.1 The theory

**Consider a transaction that provides that, in return for outlays of amount $a_{t_1}, a_{t_2}, \dots, a_{t_n}$ at time $t_1, t_2, \dots, t_n$, an investor will receive payments of $b_{t_1}, b_{t_2}, \dots, b_{t_n}$ at these times respectively. (In most situations only one of $a_{t_r}$ and $b_{t_r}$ will be non-zero.) At what force or rate of interest does the series of outlays have the same value as the series of receipts? At force of interest $\delta$ the two series are of equal value if and only if:**

$$\sum_{r=1}^{n} a_{t_r} e^{-\delta t_r} = \sum_{r=1}^{n} b_{t_r} e^{-\delta t_r}$$

**This equation may be written as:**

$$\sum_{r=1}^{n} c_{t_r} e^{-\delta t_r} = 0 \tag{1.1}$$

**where $c_{t_r} = b_{t_r} - a_{t_r}$ is the amount of the *net cashflow* at time $t_r$. (We adopt the convention that a negative cashflow corresponds to a payment *by* the investor and a positive cashflow represents a payment *to* the investor.)**

**Equation (1.1), which expresses algebraically the condition that, at force of interest $\delta$, the total value of the net cashflows is 0, is called the equation of value for the force of interest implied by the transaction. If we let $e^{\delta} = 1+i$, the equation may be written as:**

$$\sum_{r=1}^{n} c_{t_r} (1+i)^{-t_r} = 0 \tag{1.2}$$

**The latter form is known as the equation of value for the rate of interest or the 'yield equation'. Alternatively, the equation may be written as:**

$$\sum_{r=1}^{n} c_{t_r} v^{t_r} = 0$$

**In relation to continuous payment streams, if we let $\rho_1(t)$ and $\rho_2(t)$ be the rates of paying and receiving money at time $t$ respectively, we call $\rho(t) = \rho_2(t) - \rho_1(t)$ the net rate of cashflow at time $t$. The equation of value (corresponding to Equation (1.1)) for the force of interest is:**

$$\int_0^{\infty} \rho(t) e^{-\delta t} dt = 0$$

**When both discrete and continuous cashflows are present, the equation of value is:**

$$\sum_{r=1}^{n} c_{t_r} e^{-\delta t_r} + \int_0^\infty \rho(t) e^{-\delta t} dt = 0 \tag{1.3}$$

**and the equivalent yield equation is:**

$$\sum_{r=1}^{n} c_{t_r} (1+i)^{-t_r} + \int_0^\infty \rho(t)(1+i)^{-t} dt = 0 \tag{1.4}$$

**For any given transaction, Equation (1.3) may have no roots, a unique root, or several roots. If there is a unique root, $\delta_0$ say, it is known as the force of interest implied by the transaction, and the corresponding rate of interest $i_0 = e^{\delta_0} - 1$ is called the 'yield' per unit time. (Alternative terms for the yield are the 'internal rate of return' and the 'money-weighted rate of return' for the transaction.) Thus the yield is defined if and only if Equation (1.4) has precisely one root greater than –1 and, when such a root exists, it is the yield.**

The yield must be greater than –1, since $e^{\delta_0} > 0$, so $i_0 = e^{\delta_0} - 1 > -1$. A yield of –1 corresponds to a return of –100%, *ie* losing all the money originally invested.

## Question

An investor pays £100 now in order to receive £60 in 5 years' time and £60 in 10 years' time.

Calculate the annual effective rate of interest earned on this investment.

## Solution

The equation of value is:

$$100 = 60v^5 + 60v^{10}$$

This is a quadratic in $v^5$, which can be solved to give:

$$v^5 = (1+i)^{-5} = \frac{-60 \pm \sqrt{60^2 + 4 \times 60 \times 100}}{120} = 0.8844 \text{ or } -1.884$$

Rearranging, this gives $i$ to be 2.49% or –188%. Since $i$ must be greater than –1, the annual effective rate of interest is 2.49%.

**The analysis of the equation of value for a given transaction may be somewhat complex depending on the shape of the function $f(i)$ denoting the left-hand side of Equation (1.4). However, when the equation $f(i) = 0$ is such that $f$ is a monotonic function, its analysis is particularly simple. The equation has a root if and only if we can find $i_1$ and $i_2$ with $f(i_1)$ and $f(i_2)$ of opposite sign. In this case, the root is unique and lies between $i_1$ and $i_2$. By choosing $i_1$ and $i_2$ to be 'tabulated' rates sufficiently close to each other, we may determine the yield to any desired degree of accuracy.**

A monotonic function is one without any turning points, so it either always increases, or always decreases.

Having identified suitable values of $i_1$ and $i_2$, linear interpolation could be used to obtain the yield $i_0$ from these. Examples of this method are given a little later in this chapter.

**It should be noted that, after multiplication by $(1+i)^{t_0}$, Equation (1.2) takes the equivalent form:**

$$\sum_{r=1}^{n} c_{t_r}(1+i)^{t_0-t_r} = 0$$

**This slightly more general form may be called the *equation of value at time* $t_0$. It is of course directly equivalent to the original equation (which is now seen to be the equation of value at time 0). In certain problems a particular choice of $t_0$ may simplify the solution.**

**Real world examples of assets with certain cashflow streams that work as described above are outlined below.**

## 1.2 A fixed-interest security

**A body such as an industrial company, a local authority, or the government of a country may raise money by floating a loan on the stock exchange.**

This means that the organisation borrows money by issuing a loan to investors. The loan is simultaneously listed on the stock exchange so that after issue the securities can be traded on the stock exchange. This means that investors can sell their right to receive the future cashflows.

**In many instances such a loan takes the form of a *fixed-interest security*, which is issued in bonds of a stated nominal amount. The characteristic feature of such a security in its simplest form is that the holder of a bond will receive a lump sum of specified amount at some specified future time together with a series of regular level interest payments until the repayment (or 'redemption') of the lump sum.**

The regular level interest payments are referred to as coupons. Thus a zero-coupon bond has no interest payments.

**The investor has an initial negative cashflow, a single known positive cashflow on the specified future date, and a series of smaller known positive cashflows on a regular set of specified future dates.**

An investor might buy a 20-year fixed-interest security of nominal amount £10,000. This means that the face value of the loan is £10,000. The investor is unlikely to pay exactly £10,000 for this security but will pay a price that is acceptable to both parties. This may be higher or lower than £10,000. The investor will then receive a lump sum payment in 20 years' time. This lump sum is most commonly equal to the nominal amount, in this case £10,000, but could be a pre-specified amount higher or lower than this. The investor will also receive regular payments throughout the 20 years of, say, £500 *pa*. These regular payments could be made at the end of each year or half-year or at different intervals.

In the case where the regular payments are made at the end of each year, the cashflows of the investor can be represented on a timeline, as follows:

| – £price | +£500 | +£500 | +£500 | … | +£500 | +£10,500 |
|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | … | 19 | 20 |

The last payment is made up of the final regular payment (£500) and the lump sum payment (£10,000).

## 1.3 An index-linked security

Inflation is a measure of the rate of change of the price of goods and services, including salaries. High inflation implies that prices are rising quickly and low inflation implies that prices are rising slowly.

If a pair of socks costs £1, then £5 could be used to buy 5 pairs of socks. However, if inflation is high, then the cost of socks in 1 year's time might be £1.25. £5 would then only buy 4 pairs of socks.

This simple example shows how the 'purchasing power' of a given sum of money, *ie* the quantity of goods that can be bought with the money, can diminish if inflation is high. In the socks example, the annual rate of inflation is 25%.

**With a conventional fixed-interest security, the interest payments are all of the same amount. If inflationary pressures in the economy are not kept under control, the purchasing power of a given sum of money diminishes with the passage of time, significantly so when the rate of inflation is high. For this reason some investors are attracted by a security for which the actual cash amount of interest payments and of the final capital repayment are linked to an 'index' which reflects the effects of inflation.**

**Here the initial negative cashflow is followed by a series of unknown positive cashflows and a single larger unknown positive cashflow, all on specified dates. However, it is known that the amounts of the future cashflows relate to the inflation index. Hence these cashflows are said to be known in 'real' terms.**

*Real terms* means taking into account inflation, whereas *nominal* (or *money*) *terms* means ignoring inflation. For example, if your salary is rising at 5% *pa* and inflation is 7% *pa*, your salary is *falling* in real terms (as you will be able to buy less with your 'higher salary'), even though your salary is *rising* in nominal (or money) terms.

So for an index-linked bond, the cashflows are known in real terms, but are unknown in nominal (or money) terms. For a fixed-interest bond, on the other hand, the cashflows are known in nominal (or money) terms, but are unknown in real terms.

Both the regular interest payments and the final capital payment from an index-linked security are linked to the inflation index. If inflation is high, then the regular payments will rise by larger amounts than if inflation is low.

If inflation is 10% per time period and the regular interest payment after one time period is £500, then the payment after two time periods will be £550 ( $=500\times1.1$ ), and the payment after three time periods will be £605 ( $=500\times1.1^2$ ) *etc*.

Inflation is often measured by reference to an index. For example an inflation index might take values as set out in the table below.

| **Date** | 1.1.2019 | 1.1.2020 | 1.1.2021 | 1.1.2022 |
|---|---|---|---|---|
| **Index** | 100.00 | 105.00 | 108.00 | 113.00 |

Based on this, the rate of inflation during 2020 is 2.86% *pa* (*ie* $108/105-1$ ).

## Question

An investor purchased a three-year index-linked security on 1.1.2019. In return, the investor received payments at the end of each year plus a final redemption payment, all of which were increased in line with the index given in the table above. The payments would have been £600 each year and £11,000 on redemption if there had been no inflation.

Calculate the payments actually received by the investor.

## Solution

The payments received by the investor are calculated in the table below:

| Payment date | Payment amount |
|---|---|
| 1.1.2020 | $600\times\dfrac{105}{100}=£630$ |
| 1.1.2021 | $600\times\dfrac{108}{100}=£648$ |
| 1.1.2022 | $(11{,}000+600)\times\dfrac{113}{100}=£13{,}108$ |

**Note that in practice the operation of an index-linked security will be such that the cashflows do not relate to the inflation index at the time of payment, due to delays in calculating the index. It is also possible that the need of the borrower (or perhaps the investors) to know the amounts of the payments in advance may lead to the use of an index from an earlier period.**

## Question

An investor purchased a two-year index-linked security on 1.1.2020. In return, the investor received payments at the end of each year plus a final redemption payment, all of which were increased in line with the index given in the table above, with a one-year indexation lag, *ie* the index value one year before each payment is used. The payments would have been £600 each year and £11,000 on redemption if there had been no inflation.

Calculate the payments actually received by the investor.

## Solution

The payments received by the investor are calculated in the table below:

| Payment date | Payment amount |
|---|---|
| 1.1.2021 | $600 \times \frac{105}{100} = £630$ |
| 1.1.2022 | $(11{,}000 + 600) \times \frac{108}{100} = £12{,}528$ |

The one-year indexation lag means that the payment on 1 January 2021 is calculated using the index values on 1 January 2019 (one year before the date of issue of the bond) and on 1 January 2020 (one year before the date of payment).

## 1.4 Cash on deposit

**If cash is placed on deposit, the investor can choose when to disinvest and will receive interest additions during the period of investment. The interest additions will be subject to regular change as determined by the investment provider. These additions may only be known on a day-to-day basis. The amounts and timing of cashflows will therefore be unknown.**

This is describing a bank account that pays interest and allows instant access. Consider your own bank account. You can choose when to invest money, *ie* pay money in, and when to disinvest money, *ie* withdraw money. The interest you receive on your money will depend on the current interest rate and this may change with little or no notice.

## 1.5 Solving for an unknown quantity

Many problems in actuarial work can be reduced to solving an equation of value for an unknown quantity. We will look at how to do this using examples based on a hypothetical fixed-interest security which operates as follows:

### Security *S*

A price $P$ is paid (by the investor) in return for a series of interest payments of $D$ payable at the end of each of the next $n$ years and a final redemption payment of $R$ payable at the end of the $n$ years. The investor earns an annual effective rate of return of $i$.

| | | | | | |
|---|---|---|---|---|---|
| Cashflows | $-P$ | $D$ | … | $D$ | $R$<br>$D$ |
| Time | 0 | 1 | … | $n-1$ | $n$ |

The equation of value for this investment is:

$$P = Da_{\overline{n}|} + Rv^n \qquad \text{calculated at interest rate } i$$

In the remainder of this section we will consider how to solve this equation when each of the quantities $P$, $D$ or $R$, $n$ and $i$ is unknown.

### Solving for the present value (*P*)

The present value (which, in this case, represents the price) can be found using formulae we derived earlier in the course.

#### Question

Calculate $P$, given that $D=5$, $R=125$, $i=10\%$ and $n=10$.

#### Solution

The price $P$ can be calculated directly (using 10% interest):

$$P = 5a_{\overline{10}|} + 125v^{10} = 5\times 6.1446 + 125\times 0.38554 = £78.92$$

Note from this example that the equation of value holds for the values $P=78.92$, $D=5$, $R=125$, $i=10\%$ and $n=10$. We will treat these values as our reference values.

The result shown in the following question is sometimes useful.

#### Question

For Security *S*, show algebraically that if $D=iR$, then $P=R$.

## Solution

The price $P$ is given by the equation of value:

$$P = iRa_{\overline{n}|} + Rv^n \quad \text{(calculated at interest rate } i \text{)}$$

Using the formula for the annuity and simplifying gives:

$$P = iR\left(\frac{1-v^n}{i}\right) + Rv^n = R(1-v^n) + Rv^n = R$$

We can also obtain this result by general reasoning.

Suppose an investor deposits a sum of money $R$ into a bank account that pays an effective annual interest rate $i$. The investor leaves the money in the account for $n$ years. If interest is paid at the end of each year, and is withdrawn as soon as it is paid, the investor will receive interest payments of $iR$ at the end of each year and the initial deposit of $R$ will be repaid at the end of $n$ years.

Under this arrangement the cashflows the investor receives exactly match the cashflows received by investing an amount $P$ in Security $S$. Also, the rate of return obtained from the bank account will be $i$ (by definition), which is the same as the interest rate $i$ required from Security $S$.

So, investing $R$ in the bank account or $P$ in Security $S$ leads to the same cashflows and gives the same rate of return. So $P$ and $R$ must be equal.

## Question

Calculate $P$, given that $D = 10$, $R = 125$, $i = 8\%$ and $n = 10$.

## Solution

The value of $P$ is:

$$P = 10a_{\overline{10}|} + 125v^{10} = 10 \times 6.7101 + 125 \times 0.46319 = £125.00$$

*This calculation verifies the result just proved, since here $D = 10 = 0.08 \times 125 = iR$ and we find that $P = 125 = R$.*

## Solving for the amount of a payment (*D* or *R*)

Solving the equation of value for $D$ or $R$ is straightforward.

## Question

Calculate $D$, given that $P = 127.12$, $R = 125$, $i = 7.75\%$ and $n = 10$.

## Solution

The equation of value is:

$$127.12 = Da_{\overline{10|}} + 125v^{10}$$

So: $$127.12 = D \times \frac{1-1.0775^{-10}}{0.0775} + 125 \times 1.0775^{-10}$$

*ie* $$127.12 = D \times 6.7864 + 125 \times 0.47405$$

This can be rearranged to find $D$:

$$D = \frac{127.12 - 125 \times 0.47405}{6.7864} = 10.00$$

## Solving for the timing of a payment (*n*)

We can solve the equation of value for $n$ by expressing the annuity function in terms of $v$.

## Question

Calculate $n$, given that $P = 78.92$, $D = 5$, $R = 125$ and $i = 0.10$.

## Solution

The equation of value is:

$$78.92 = 5a_{\overline{n|}} + 125v^n$$

Substituting the formula for $a_{\overline{n|}}$ gives:

$$78.92 = 5 \times \frac{1-v^n}{0.10} + 125v^n$$

*ie* $$78.92 = 50(1-v^n) + 125v^n = 50 + 75v^n$$

This can be rearranged to find $v^n$:

$$v^n = \frac{78.92-50}{75} = 0.38560$$

*ie* $$1.10^{-n} = 0.38560$$

Taking logs, and using the result $\log a^b = b\log a$, we find:

$$-n\log 1.10 = \log 0.38560 \quad ie \quad n = -\frac{\log 0.38560}{\log 1.10} = 10.00$$

## Solving for the interest rate (*i*)

Finding the interest rate is the hardest type of calculation, since the equation of value cannot usually be solved explicitly. If the equation of value cannot be solved explicitly, we could use trial and error, based on a rough initial guess.

To obtain an initial guess, we can approximate the interest rate by combining the cashflows into a single payment, payable on an average payment date. This is illustrated in the following question.

### Question

Given that $P = 78.92$, $D = 5$, $R = 125$ and $n = 10$, determine a rough estimate for the value of $i$.

### Solution

The equation of value is:

$$78.92 = 5a_{\overline{10|}} + 125v^{10} \quad \text{(calculated at interest rate } i\text{)}$$

Here, a payment of 5 is received at the end of each of years 1 to 10 (roughly equivalent to a total of 50 paid on average at time 5½), and in addition a payment of 125 is received at the end of year 10. Combining these (and weighting the timings by amounts) gives a single payment of 175 (*ie* $50 + 125$) at time 8.7 (*ie* $(50 \times 5½ + 125 \times 10)/175$). This gives an equation we can solve more easily:

$$78.92 \approx 175v^{8.7} \text{(calculated at rate } i\text{)}$$

So:

$$1 + i \approx \left(\frac{78.92}{175}\right)^{-\frac{1}{8.7}} = 1.096 \quad ie \quad i \approx 9.6\%$$

This rough estimate is quite close to the exact value of 10% (which we know from earlier questions).

An alternative method for finding a first guess is to use a first-order binomial expansion, replacing $(1+i)^n$ by $(1+ni)$. However, this is better suited to equations of value that contain no annuities. For example, using this method here we would have attained a first guess of 7.7%:

$$78.92 = 5\left(\frac{1-(1+i)^{-10}}{i}\right) + 125(1+i)^{-10}$$

$$\Rightarrow \quad 78.92 \approx 5\left(\frac{1-(1-10i)}{i}\right) + 125(1-10i)$$

$$\Rightarrow \quad 78.92 \approx 50 + 125(1-10i)$$

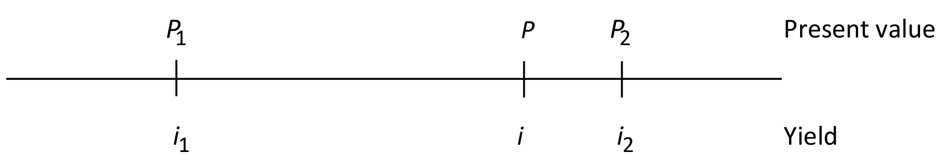
$$\Rightarrow \quad i \approx \frac{96.08}{1{,}250} = 7.7\%$$

Once an initial estimate has been obtained, a more accurate solution can then be found from the exact equation by linear interpolation, using the initial estimate as a starting point.

## Estimating an unknown interest rate using linear interpolation

Suppose that the present values, calculated at interest rates $i_1$ and $i_2$, are $P_1$ and $P_2$ respectively, and we wish to work out the approximate interest rate corresponding to a present value of $P$.

This situation is illustrated on the following diagram




If the present value is a linear function of the interest rate, then the proportionate change in the interest rates will equal the proportionate change in the present values:

$$\frac{i-i_1}{i_2-i_1} = \frac{P-P_1}{P_2-P_1}$$

Rearranging this relationship gives the approximate value of $i$:

$$i \approx i_1 + \frac{P-P_1}{P_2-P_1} \times (i_2 - i_1)$$

This approximation works best if the trial values are close to the true value, *eg* values that are 1% apart. This formula also works even if the true value does not lie *between* the two trial values, but we would not recommend this approach (*ie* extrapolation) in the exam. We recommend that you interpolate between values that are either side of the true value and are a maximum of 1% apart. It is even better if the two values are 0.5% apart.

## Question

Given that $P=75$, $D=5$, $R=125$ and $n=10$, calculate the value of $i$.

## Solution

The equation of value is:

$$75=5a_{\overline{10|}}+125v^{10}$$

We know that when $i=10\%$, the right-hand side of this equation is equal to 78.92.

The price paid (75) is lower than this. So the value of $i$ must be greater than 10%. Using $i=11\%$, the right-hand side is:

$$5a_{\overline{10|}}+125v^{10}=5\left(\frac{1-1.11^{-10}}{0.11}\right)+125(1.11)^{-10}=73.47$$

Interpolating linearly using these two values gives:

$$i\approx 10\%+\frac{75-78.92}{73.47-78.92}\times(11\%-10\%)=10.7\%$$

## 1.6 Example applications

Later in the course we will use equations of value in the context of the most common types of investment: fixed-interest bonds, index-linked bonds, equities (*ie* shares) and property.

Here, as an example, we will look at a question involving a property investment.

## Question

A company has just bought an office block for £5m, which it will rent out to a number of small businesses. The total rent for the first year will be £100,000, and this is expected to increase by 4% *pa* compound in each future year. The office block is expected to be sold after 20 years for £7.5m.

Assuming that rent is paid in the middle of each year, calculate the yield the company will obtain on this investment.

## Solution

Working in £000s, the equation of value here is:

$$5,000 = 100(v^{½} + 1.04v^{1½} + 1.04^2 v^{2½} + \cdots + 1.04^{19} v^{19.5}) + 7,500v^{20}$$

The terms in brackets form a geometric progression of 20 terms, with first term $v^{½}$ and common ratio $1.04v$. So the equation of value can be written:

$$5,000 = \frac{100v^{½}\left(1-(1.04v)^{20}\right)}{1-1.04v} + 7,500v^{20} \qquad \text{provided } v \neq \frac{1}{1.04}$$

We can solve this by trial and error. At 4%, the right-hand side (using the first expression) is:

$$100\left(1.04^{-½} + 1.04^{-½} + \cdots + 1.04^{-½}\right) + \frac{7,500}{1.04^{20}} = 100 \times 20 \times 1.04^{-½} + \frac{7,500}{1.04^{20}} = 5,384.06$$

At 5%, the right-hand side (using the second expression) is 4,611.57. Interpolating between these two values, we obtain:

$$i \approx 4\% + \frac{5,000 - 5,384.06}{4,611.57 - 5,384.06} \times (5\% - 4\%) = 4.5\%$$

# 2 Uncertain payment or receipt

**If there is uncertainty about the payment or receipt of a cashflow at a particular time, allowance can be made in one of two ways:**

- **apply a probability of payment/receipt to the cashflow at each time, or**
- **use a higher rate of discount.**

## 2.1 Probability of cashflow

**The probability of payment/receipt can be allowed for by adapting the earlier equations. For example, Equation (1.4) can be revised to produce:**

$$\sum_{r=1}^{n} p_{t_r} c_{t_r} (1+i)^{-t_r} + \int_0^{\infty} p(t)\rho(t)(1+i)^{-t}\,dt = 0 \qquad (2.1)$$

**where $p_t$ and $p(t)$ represent the probability of a cashflow at time $t$.**

**Where the force of interest is constant, and we can say that the probability is itself in the form of a discounting function, then Equation (1.3) can be generalised as:**

$$\sum_{r=1}^{n} c_{t_r} e^{-\delta t_r} e^{-\mu t_r} + \int_0^{\infty} \rho(t) e^{-\delta t} e^{-\mu t}\,dt = 0 \qquad (2.2)$$

**where $\mu$ is a constant force, rather than rate, of the probability of a cashflow at time $t$.**

**These probabilities of cashflows may often be estimated by consideration of the experience of similar cashflows. For example, this approach is used to assess the probabilities of cashflows that are dependent on the survival of a life – this is the theme of later chapters.**

**In other cases, there may be lack of data from which to determine an accurate probability for a cashflow. Instead a more approximate probability, or likelihood, may be determined after careful consideration of the risks.**

**In some cases, it may be spurious to attempt to determine the probability of each cashflow and so more approximate methods may be justified.**

**Wherever the uncertainty about the probability of the amount or timing of a cashflow could have significant financial effect, a sensitivity analysis may be performed. This involves calculations performed using different possible values for the likelihood and the amounts of the cashflows. Alternatively, a stochastic approach could be used to indicate possible outcomes.**

## Question

A lottery ticket costs £1. The following table shows the different prizes available together with the chance of winning and the delay before receiving the prize money.

| Prize | Probability of winning | Time before payment |
|---|---|---|
| £20 | 1 in 50 | 1 day |
| £200 | 1 in 1,000 | 1 day |
| £2,000 | 1 in 50,000 | 1 week |
| £200,000 | 1 in 2 million | 2 weeks |
| £2 million | 1 in 14 million | 4 weeks |

Calculate the expectation of the present value of the prize money assuming an effective rate of interest of 0.016% per day.

## Solution

To calculate the expectation of the present value of the prize money, we take each possible present value, multiply it by the probability that it occurs, and then sum over all possibilities.

So, working in days, the expectation of the present value of the prize money is:

$$\frac{1}{50}\times 20v+\frac{1}{1{,}000}\times 200v+\frac{1}{50{,}000}\times 2{,}000v^{7}$$
$$+\frac{1}{2{,}000{,}000}\times 200{,}000v^{14}+\frac{1}{14{,}000{,}000}\times 2{,}000{,}000v^{28}$$
$$=£0.88$$

**It is important to understand the different approaches to modelling and what is meant by sensitivity analysis. A brief outline of these topics is included below.**

## 2.2 Stochastic and deterministic models

**If it is desired to represent reality as accurately as possible, the model needs to imitate the random nature of the variables. A *stochastic* model is one that recognises the random nature of the input components. A model that does not contain any random component is *deterministic* in nature.**

**In a deterministic model, the output is determined once the set of fixed inputs and the relationships between them have been defined. By contrast, in a stochastic model the output is random in nature – like the inputs, which are random variables. The output is only a snapshot or an estimate of the characteristics of the model for a given set of inputs. Several independent runs are required for each set of inputs so that statistical theory can be used to help in the study of the implications of the set of inputs.**

**A deterministic model is really just a special (simplified) case of a stochastic model.**

The following example illustrates the difference between the two approaches.

## Question

An investor has bought shares worth £5,000 and wants to estimate how much they will be worth in a year's time. Describe both a deterministic and stochastic model based on an expected growth rate of 7% over the year.

## Solution

***Deterministic model***

The outcome from a deterministic model is the prediction that the value in a year's time would be:

$$5{,}000 \times 1.07 = £5{,}350$$

***Stochastic model***

A stochastic model allows for randomness in the growth rate. For example, it might be decided (based on past performance of the company and prospects for the company, the investment sector and the economy in general) that the probabilities of different growth rates for the shares are:

$$\text{growth} = \begin{cases} 20\% & \text{with probability } 0.1 \\ 10\% & \text{with probability } 0.6 \\ 0\% & \text{with probability } 0.2 \\ -10\% & \text{with probability } 0.1 \end{cases}$$

These probabilities add up to 1 and the expected growth rate is 7% for this model since:

$$0.1 \times 20\% + 0.6 \times 10\% + 0.2 \times 0\% + 0.1 \times (-10\%) = 7\%$$

The outcome from this model, if it is run 100 times, is a list of 100 predicted values, which might look like this:

£5500, £5000, £5000, £6000, £5000, … , £4500, £6000

**Whether one wishes to use a deterministic or a stochastic model depends on whether one is interested in the results of a single 'scenario' or in the distribution of results of possible 'scenarios'. A deterministic model will give one the results of the relevant calculations for a single scenario; a stochastic model gives distributions of the relevant results for a distribution of scenarios. If the stochastic model is investigated by using 'Monte Carlo' simulation, then this provides a collection of a suitably large number of different deterministic models, each of which is considered equally likely.**

'Monte Carlo' simulation is where a computer is set up to run a stochastic model a number of times, *eg* 10,000 times, using pseudo-random numbers generated by the computer. The results look like the list of numbers in the stochastic model in the example above. The numbers generated by the computer are pseudo-random rather than truly random because they are generated using a prescribed method.

As stochastic models provide a distribution of outputs, they can be used to estimate the probability that a particular event occurs, or to calculate approximate confidence intervals. This wouldn't be possible using a deterministic model.

**The results for a deterministic model can often be obtained by direct calculation, but sometimes it is necessary to use numerical approximations, either to integrate functions or to solve differential equations.**

**If a stochastic model is sufficiently tractable, it may be possible to derive the results one wishes by analytical methods. If this can be done it is often preferable to, and also often much quicker than, Monte Carlo simulation. One gets precise results and can analyse the effect of changes in the assumptions more readily. Monte Carlo simulation is covered in CS1.**

**Many practical problems, however, are too complicated for analytical methods to be used easily, and Monte Carlo simulation is an extremely powerful method for solving complicated problems. But if even part of a model can be treated analytically, it may provide a check on any simulation method used. It may be possible to use a deterministic method to calculate the expected values, or possibly the median values, for a complicated problem, where the distributions around these central values are estimated by simulation.**

**One also needs to recognise that a simulation method generally provides 'what if?' answers; what the results are on the basis of the assumptions that have been made. It is much harder to use simulation to provide the optimum solution; in other words to find the set of assumptions that maximises or minimises some desired result.**

**A further limitation is that the precision of a simulated result depends on the number of simulations that are carried out. This is covered in more detail in CS1.**

## 2.3 Sensitivity testing

**Where possible, it is important to test the reasonableness of the output from the model against the real world. To do this, an examination of the sensitivity of the outputs to small changes in the inputs or their statistical distributions should be carried out. The appropriateness of the model should then be reviewed, particularly if small changes in inputs or their statistical distributions give rise to large changes in the outputs. In this way, the key inputs and relationships to which particular attention should be given in designing and using the model can be determined.**

If small changes in the inputs give rise to large changes in the outputs, then our initial choices are more crucial. How confident are we in our choices of input? If the resulting changes in output are small, then our initial choices are less important in this respect.

Sensitivity testing also refers to the approach of using a deterministic model with changes in one or more of the assumptions to see the range of possible outcomes that might occur. For example, an insurance company providing personal pension plans might give illustrations to policyholders showing how much pension they would get if growth rates were 2%, 5% and 8% *pa* in the period before retirement. This allows the policyholder to gauge the extent to which the resulting pension will be affected by changes in the growth rate.

**The model should be tested by designing appropriate simulation experiments. It is through this process that the model can be refined.**

An approach that has been traditionally used by actuaries in the fields of insurance, pensions and investment is to carry out a set of deterministic calculations using different actuarial bases, *ie* under different sets of assumptions. By varying the assumptions, the actuary could use the model to arrive at figures that are 'best estimates' (the most likely, or median, result) or 'optimistic' (if things work out favourably) or 'cautious' (if things work out badly). This is an example of a scenario-based approach to modelling.

## 2.4 Higher discount rate

**As the discounting functions and the probability functions in Equations (2.1) and (2.2) are both dependent on time, they can be combined into a single time-dependent function. In cases where there is insufficient information to objectively produce the probability functions, this combined function can be viewed as an adjusted discounting function that makes an implicit allowance for the probability of the cashflow.**

**Where the probability of the cashflow is a function that is of similar form to the discounting function, the combination can be treated as if a different discount rate were being used. For example, Equation (2.2) becomes:**

$$\sum_{r=1}^{n} c_{t_r} e^{-\delta' t_r} + \int_0^{\infty} \rho(t) e^{-\delta' t} dt = 0$$

**where** $\delta' = \delta + \mu$**.**

**The revised force of discount is therefore greater than the actual force of discount, as** $\mu$ **must be positive in order to give a probability between 0 and 1. It can therefore be shown that the rate of discount that is effectively used is greater than the actual rate of discount before the implicit allowance for the probability of the cashflow.**

## Question

An individual has invested in a company run by some ex-criminals. In return for the investment they expect to receive £100 at the end of each of the next ten years. The annual effective interest rate is 5%.

Calculate the present value of their investment by:

(i) ignoring the possibility that the payments might not be made.

(ii) assuming the probability of receiving the first payment is 0.95, the second payment is 0.9, the third payment is 0.85 *etc*.

(iii) increasing the force of interest by 0.04652.

## Solution

(i) $PV = 100a_{\overline{10}|}^{@5\%} = 100 \times 7.7217 = £772.17$

(ii) The present value allowing for the probabilities of payment is:

$$PV = 100v \times 0.95 + 100v^2 \times 0.9 + \cdots + 100v^{10} \times 0.5$$

$$= 95v + 90v^2 + \cdots + 50v^{10}$$

This is equivalent to an annuity in arrears with decreases of 5 each year. So the present value is:

$$PV = 100a_{\overline{10}|} - 5(Ia)_{\overline{10}|} = 100 \times 7.7217 - 5 \times 39.3738 = £575.30$$

(iii) The new force of interest is $\ln(1.05) + 0.04652 = \ln(1.1)$. Therefore we can use an effective rate of interest of 10% *pa*:

$$PV = 100a_{\overline{10}|}^{@10\%} = 100 \times 6.1446 = £614.46$$

The chapter summary starts on the next page so that you can
keep all the chapter summaries together for revision purposes

## Chapter 7 Summary

An equation of value equates the present value of money received to the present value of money paid out:

$$PV\ income - PV\ outgo = 0$$

$$\sum_{r=1}^{n} c_{t_r} e^{-\delta t_r} + \int_0^\infty \rho(t) e^{-\delta t} dt = 0$$

$$\sum_{r=1}^{n} c_{t_r} (1+i)^{-t_r} + \int_0^\infty \rho(t)(1+i)^{-t} dt = 0$$

To calculate the yield on a transaction from an equation of value of the form $f(i)=0$, we need:

- to find $i_1$ and $i_2$ such that $f(i_1)$ and $f(i_2)$ are of opposite sign, and
- the final value obtained for $i$ to be greater than $-1$.

Some equations of value cannot be solved algebraically. In such cases, we might calculate the yield using a trial and error approach, in conjunction with linear interpolation. The formula for linear interpolation is:

$$i \approx i_1 + \frac{P - P_1}{P_2 - P_1} \times (i_2 - i_1)$$

If there is uncertainty about the payment or receipt of a cashflow at a particular time, allowance can be made in one of two ways:

- apply a probability of payment/receipt to the cashflow at each time
- use a higher rate of discount such as a new force of interest $\delta'$, where $\delta' = \delta + \mu$.

A model may be stochastic or deterministic. For stochastic models it is better to use direct calculation if possible, but in complex situations it may be necessary to use Monte Carlo simulation on a computer.

The practice questions start on the next page so that you can keep the chapter summaries together for revision purposes.

# Chapter 7 Practice Questions

Questions 7.1 to 7.3 relate to financial security *S*, which operates as follows:

A price $P$ is paid (by the investor) in return for a series of interest payments of $D$ payable at the end of each of the next $n$ years, and a final redemption payment of $R$ payable at the end of the $n$ years. The investor earns an annual effective rate of return of $i$.

7.1 Calculate $P$, given that $D=5$, $R=125$, $i=10\%$ and $n=20$.

7.2 Calculate $n$, given that $P=83.73$, $D=4$, $R=101$ and $i=6\%$.

7.3 Calculate $i$, given that $P=75$, $D=5$, $R=120$ and $n=10$.

7.4 An investor is to pay £80,000 for a property. The investor will then be entitled to receive rental payments at the end of each year for 99 years. The rental payment will be fixed for the first 33 years, increasing to double the original amount for the next 33 years, and three times the original amount for the remaining 33 years. The value of the property at the end of the 99 years is expected to be £1,500,000.

Calculate the amount of the rent payable in the first year, if the investor expects to obtain a rate of return of 8% *pa* effective on the purchase.

7.5 An investor deposits £2,000 into an account, then withdraws level annual payments starting one year after the deposit is made. Immediately after the 11th annual withdrawal, the investor has £400 left in the account. Calculate the amount of each withdrawal, given that the effective annual rate of interest is 8%.

The solutions start on the next page so that you can separate the questions and solutions.

# Chapter 7 Solutions

7.1 The price $P$ can be calculated directly as follows:

$$P = 5a_{\overline{20}|} + 125v^{20} = 5 \times 8.5136 + 125 \times 0.14864 = 61.15$$

7.2 The term $n$ satisfies the equation:

$$83.73 = 4a_{\overline{n}|} + 101v^n = 4\left(\frac{1-1.06^{-n}}{0.06}\right) + 101 \times 1.06^{-n}$$

Rearranging gives:

$$\left(101 - \frac{4}{0.06}\right)1.06^{-n} = 83.73 - \frac{4}{0.06} \quad \Rightarrow \quad 1.06^{-n} = 0.49699$$

Taking logs, we find:

$$-n\ln 1.06 = \ln 0.49699 \quad \Rightarrow \quad n = -\frac{\ln 0.49699}{\ln 1.06} = 12.00 \text{ years}$$

7.3 To find the yield, we must solve the equation of value:

$$75 = 5a_{\overline{10}|} + 120v^{10}$$

At 10%, RHS = 76.99.

At 11%, RHS = 71.71.

Interpolating, we find that $i \approx 0.10 + \dfrac{76.99-75}{76.99-71.71}(0.11-0.10) = 0.1038$.

So $i$ is approximately 10.4% *pa*.

7.4 If the amount of the rent payable in the first year is $X$, the equation of value is:

$$80,000 = Xa_{\overline{33}|} + 2Xv^{33}a_{\overline{33}|} + 3Xv^{66}a_{\overline{33}|} + 1,500,000v^{99}$$

*ie* $$80,000 = X(1 + 2v^{33} + 3v^{66})a_{\overline{33}|} + 1,500,000v^{99}$$

So:

$$80,000 = X(1 + 2 \times 1.08^{-33} + 3 \times 1.08^{-66}) \times 11.5139 + 1,500,000 \times 1.08^{-99}$$

This can be rearranged to find $X$:

$$X = \frac{80,000 - 736.444}{13.545} = £5,852$$

7.5 If the amount of the annual withdrawal is $X$, then we need to solve the equation:

$$2{,}000 = Xa_{\overline{11}|} + 400v^{11}$$

So:

$$2{,}000 = 7.1390X + 400 \times 1.08^{-11}$$

Rearranging to find $X$ gives:

$$X = \frac{1{,}828.45}{7.1390} = £256.12$$

# End of Part 1

## What next?

1. Briefly **review** the key areas of Part 1 and/or re-read the **summaries** at the end of Chapters 1 to 7.
2. Ensure you have attempted some of the **Practice Questions** at the end of each chapter in Part 1. If you don't have time to do them all, you could save the remainder for use as part of your revision.
3. Attempt **Assignment X1**.
4. Attempt the questions relating to Chapters 1 to 7 of the **Paper B Online Resources (PBOR)**.

**X1.1** Calculate the present value of a perpetuity paying £50 *pa* in arrears at an annual effective rate of interest of 6%. [1]

**X1.2** Calculate the length of time it will take £800 to accumulate to £1,000 at a simple rate of interest of 4% *pa*. [2]

**X1.3** Explain the main differences between a deterministic model and a stochastic model. [3]

**X1.4** A 91-day government bill is discounted at a simple rate of discount of 10% *pa*. Calculate the annual effective rate of interest earned on this investment. [3]

**X1.5** An annuity of $300 *pa* is paid annually in advance for seven years, followed by $100 *pa* paid quarterly in arrears for a further five years. The rate of interest is 6% *pa* convertible half-yearly.

Determine the accumulated amount at the end of twelve years. [5]

**X1.6** Calculate, at a rate of interest of 8% per annum effective:

(a) $a_{\overline{5}|}^{(4)}$ [1]

(b) ${}_{2|}\ddot{a}_{\overline{15}|}$ [1]

(c) $(I\bar{a})_{\overline{10}|}$ [2]

(d) $(\bar{I}\bar{a})_{\overline{10}|}$ [2]

[Total 6]

**X1.7** An investor, who has a sum of £10,000 to invest, wishes to purchase an annuity certain which makes payments over a ten-year period. Calculate the amount of the payments that can be provided if the annuity takes each of the following forms (assuming interest of 8% *pa* effective):

(i) a level annuity payable monthly in arrears [2]

(ii) a level annuity due payable half-yearly, commencing in 2 years' time. [2]

[Total 4]

**X1.8** Explain when you would use real and money rates of interest. Give an example of when each rate of interest would be used. [4]

**X1.9** (i) Calculate the nominal annual rate of discount convertible quarterly equivalent to a nominal rate of interest of 10% *pa* convertible quarterly. [2]

(ii) A single investment of £500 is accumulated at a nominal rate of discount of 6% *pa* convertible half-yearly for 1 year, followed by a nominal rate of interest of 6% *pa* convertible every 4 months for 1 year. Calculate the accumulated amount of this investment after 2 years. [3]

[Total 5]

**X1.10** For each of the following calculate the equivalent effective annual rate of interest:

(i) an effective rate of interest of 12.7% paid every 2 years [1]

(ii) an effective rate of discount of 5.75% *pa* [1]

(iii) a force of interest of ½% per month [1]

(iv) a nominal rate of discount of 6% *pa* convertible quarterly [1]

(v) a nominal rate of interest of 14% *pa* convertible every 2 years. [1]

[Total 5]

**X1.11** An investor receives payments half-yearly in arrears for 20 years. The first payment is £250, and each payment is 2% higher than the previous one.

The interest rate is 6% *pa* effective for the first 10 years and 4% *pa* effective for the final 10 years.

Calculate, showing all workings, the present value of the payments. [7]

**X1.12** The force of interest, $\delta(t)$, is a function of time and at time $t$, measured in years, is given by:

$$\delta(t) = 0.03 - 0.005t + 0.001t^2 \qquad 0 \le t \le 10$$

(i) Calculate the equivalent constant force of interest per annum for the period $t = 0$ to $t = 10$. [3]

(ii) Calculate, showing all workings, the accumulated value at time $t = 7$ of an investment of £250 at time $t = 0$ plus a further investment of £150 at time $t = 5$. [5]

[Total 8]

**X1.13** An annuity payable monthly in arrears has a first payment of £300, with subsequent payments decreasing by £10 each month, until a final payment of £70 is made in two years' time.

Calculate the present value of the payments from this annuity using an effective rate of interest of 6% *pa.* [4]

**X1.14** In return for a fixed initial deposit, an investor receives a continuously payable annuity for a term of 15 years. The annual rate of payment is 50 in the first year, and the rate of payment increases in each subsequent year.

The investor can select either:

- Option 1: the rate of payment increases by 2 at the end of each year,
- Option 2: the rate of payment increases by 3.5% *pa* compound at the end of each year.

Determine which option would provide the better deal for the investor at an annual effective interest rate of 7%. [8]

**X1.15** The force of interest $\delta(t)$ is a function of time, and at any time $t$, measured in years, is given by the formula:

$$\delta(t) = \begin{cases} 0.04 + 0.005t & 0 \le t < 6 \\ 0.16 - 0.015t & 6 \le t < 8 \\ 0.04 & 8 \le t \end{cases}$$

(i) Derive expressions in terms of $t$ for the accumulated amount at time $t$ of an investment of 1 at time 0. [6]

(ii) (a) Calculate the value at time 0 of £100 due at time 9.

(b) Calculate the annual effective rate of discount implied by the transaction in (a). [3]

(iii) A continuous payment stream is received at a rate of $45e^{0.01t}$ units per annum between time 10 and time 15. Calculate, showing all workings, the present value at time 4 of this payment stream. [6]

[Total 15]

**END OF PAPER**

# Assignment X1 – Solutions

*Markers: This document sets out one approach to solving each of the questions. Please give credit for other valid approaches. In particular, in 'trial and error' questions, full marks should be awarded for obtaining the correct final answer whatever method is used (eg 'table mode' on a calculator), so long as sufficient working is given.*

| |
|---|
| Additional solutions to mathematical parts of questions are provided in boxes in order to illustrate how they could be answered in the exam, using Word. Other presentations and ways of typing out the notation would also be acceptable. |

## Solution X1.1

| |
|---|
| *Perpetuities are covered in Chapter 5, Section 6.* |

$$PV = 50a_{\overline{\infty}|} = \frac{50}{i} = £833.33$$ [1]

[Total 1]

| |
|---|
| PV = 50 * a:<infinity> @6% = 50 / 0.06 = £833.33 |

## Solution X1.2

| |
|---|
| *Simple interest is covered in Chapter 1, Section 1.* |

The length of time can be found from the equation:

$$800(1+0.04t) = 1{,}000$$ [1]

$$\Rightarrow \quad t = 6\tfrac{1}{4}\text{years}$$ [1]

[Total 2]

| |
|---|
| Need to find n that solves: |
| 800 * (1 + n*0.04) = 1,000 |
| This makes n = 6.25 years |

## Solution X1.3

*Deterministic and stochastic models are introduced in Chapter 7, Section 2.*

A deterministic model uses one set of input parameters and gives the results of the relevant calculations for this single scenario. [1]

A stochastic model involves at least one input parameter varying according to an assumed probability distribution. As such, the output will vary along with the input and the model produces distributions of the relevant results for a distribution of scenarios. [1]

Often the output from a stochastic model is in the form of many thousands of simulated outcomes of the process. We can study the distributions of these outcomes. [1]

[Total 3]

## Solution X1.4

*Simple discount is covered in Chapter 1, Section 3.*

The discount factor for 91 days under simple discount is:

$$(1-nd)=\left(1-\tfrac{91}{365}\times 0.10\right)=0.9751 \qquad [1]$$

Equating this to the effective discount factor:

$$v^{\frac{91}{365}}=0.975 \quad \Rightarrow \quad (1+i)^{-\frac{91}{365}}=0.975 \qquad [1]$$

$$\Rightarrow \quad (1+i)=0.975^{-\frac{365}{91}} \Rightarrow \quad i=10.66\% \qquad [1]$$

[Total 3]

*Alternatively, using $\frac{91}{365.25}$ or even $\frac{1}{4}$ (since 91 days is approximately a quarter of a year) in both calculations will give the same answer to 4 SF.*

We need to solve:

(1+i)^-(91/365) = (1 – (91/365) * 0.1) = 0.9751

So i = 10.66%

## Solution X1.5

*Accumulations of level annuities are covered in Chapter 5, Sections 2 and 4.*

The annual rate of interest, $i$, is given by:

$$(1+i)=1.03^2=1.0609 \quad\Rightarrow\quad i=6.09\% \qquad [½]$$

The present value of the annuity is given by:

$$PV=300\ddot{a}_{\overline{7}|}+100v^7 a_{\overline{5}|}^{(4)} \qquad [1]$$

Now:

$$\ddot{a}_{\overline{7}|}=\frac{1-v^7}{d}=\frac{1-1.0609^{-7}}{1-1.0609^{-1}}=5.90345 \qquad [½]$$

and:

$$a_{\overline{5}|}^{(4)}=\frac{1-v^5}{i^{(4)}}=\frac{1-1.0609^{-5}}{4\left[1.0609^{¼}-1\right]}=\frac{1-1.0609^{-5}}{0.059557}=4.29685 \qquad [½]$$

So:

$$PV=300\times 5.90345+100\times 1.0609^{-7}\times 4.29685 \qquad [½]$$

$$=1771.04+284.07$$

$$=\$2,055.11 \qquad [1]$$

So the accumulated amount after 12 years is:

$$\$2,055.11\times 1.0609^{12}=\$4,178 \qquad [1]$$

[Total 5]

*Alternatively, this could be calculated as:*

$$300\ddot{s}_{\overline{7}|}(1+i)^5+100s_{\overline{5}|}^{(4)}$$

*We could calculate the second annuity by working in quarters eg* $25v^7_{6.09\%}a_{\overline{20}|\,@1.4889157\%}$ *or* $25s_{\overline{20}|\,@1.4889157\%}$.

Accumulated value at time 12 is:

AV = 300 * adue:<7> * (1+i)^12 + 100 * s(4):<5>

where:

adue:<7> = (1 – (1+i)^-7) / d

i = 1.03^2 – 1 = 0.0609

d= i / 1+i

So:

adue:<7> = 5.90345

Also:

s(4):<5 = ((1+i)^5 – 1) / i(4)

where: i(4) = (1.03^0.5 – 1) * 4 = 0.059557

So:

s(4):<5> = 5.77461

So:

AV = 300 * 5.90345 * 1.0609^12 + 100 * 5.77461 = $4,177.61

## Solution X1.6

*Level annuities payable pthly are covered in Chapter 5, Section 4. Deferred annuities are covered in Chapter 5, Section 7. Increasing, continuously payable annuities are covered in Chapter 6, Section 1.*

(a) $a_{\overline{5}|}^{(4)}$

$$a_{\overline{5}|}^{(4)} = \frac{1-v^5}{i^{(4)}} = \frac{1-1.08^{-5}}{4\left[(1.08)^{1/4}-1\right]} = \frac{1-1.08^{-5}}{0.077706} = 4.110571 \qquad [1]$$

a(4):<5> = (1 – v^5 / i(4)) = (1 – 1.08^-5) / 0.077706 = 4.110571

(b) ${}_{2|}\ddot{a}_{\overline{15}|}$

$${}_{2|}\ddot{a}_{\overline{15}|} = v^2 \times \ddot{a}_{\overline{15}|} = 1.08^{-2} \times \frac{1-1.08^{-15}}{0.08/1.08} = 7.925443 \qquad [1]$$

2|adue:<15> =v^2 * (1 – v^15) / d = 1.08^-2 * (1 – 1.08^-15) / (0.08 / 1.08) = 7.925443

(c) $(I\bar{a})_{\overline{10}|}$

$$(I\bar{a})_{\overline{10}|} = \frac{\ddot{a}_{\overline{10}|} - 10v^{10}}{\delta}$$ [½]

$$\ddot{a}_{\overline{10}|} = \frac{1-1.08^{-10}}{0.08/1.08} = 7.246888$$ [½]

$$(I\bar{a})_{\overline{10}|} = \frac{7.246888 - 10 \times 1.08^{-10}}{\ln(1.08)} = 33.977621$$ [1]

(Iabar):<10> = (adue:<10> – 10*v^10) / delta

adue:<10> = (1 – 1.08^-10) / (0.08 / 1.08) = 7.246888

(Iabar):<10> = (7.246888 – 10*1.08^-10) / ln(1.08) = 33.977621

(d) $(\bar{I}\bar{a})_{\overline{10}|}$

$$(\bar{I}\bar{a})_{\overline{10}|} = \frac{\bar{a}_{\overline{10}|} - 10v^{10}}{\delta}$$ [½]

$$\bar{a}_{\overline{10}|} = \frac{1-1.08^{-10}}{\ln(1.08)} = 6.975042$$ [½]

$$(\bar{I}\bar{a})_{\overline{10}|} = \frac{6.975042 - 10 \times 1.08^{-10}}{\ln(1.08)} = 30.445370$$ [1]

[Total 6]

(Ibarabar):<10> = (abar:<10> – 10*v^10) / delta

abar:<10> = (1 – 1.08^-10) / ln(1.08) = 6.975042

(Ibarabar):<10> = (6.975042 – 10*1.08^-10) / ln(1.08) = 30.445370

## Solution X1.7

*Level annuities payable pthly are covered in Chapter 5, Section 4.*

(i) ***Level monthly annuity***

Let $X$ be the monthly amount paid. Then the present value of the payments is:

$$12Xa_{\overline{10}|}^{(12)} = 12X\left(\frac{1-1.08^{-10}}{i^{(12)}}\right) = 83.4324X \qquad [1]$$

where $i^{(12)} = 12\left(1.08^{1/12} - 1\right) = 7.7208\%$.

Since the investor paid £10,000, $X = \dfrac{10{,}000}{83.4324} = £119.86$. [1]

[Total 2]

*Alternatively, we could work in months:* $Xa_{\overline{120}|@0.643403011\%}$.

Need to find X where:

10,000 = 12 * X * a(12):<10>

and:

a(12):<10> = (1 – 1.08^-10) / i(12) = (1 – 1.08^-10) / 0.077208 = 6.95273

i(12) is taken from the Tables.

So: X = £119.86

(ii) ***Level annuity payable half yearly***

Let $Y$ be the half-yearly amount. Then the present value of the payments is:

$$2Yv^2\ddot{a}_{\overline{10}|}^{(2)} = 2Y\left(\frac{1}{1.08^2}\right)\left(\frac{1-1.08^{-10}}{d^{(2)}}\right) = 12.19154Y \qquad [1]$$

where $d^{(2)} = 2\left(1 - 1.08^{-1/2}\right) = 7.5499\%$.

Since the investor paid £10,000, $Y = \dfrac{10{,}000}{12.19154} = £820.24$. [1]

[Total 2]

*Alternatively, we could calculate the annuity by working in half-years:* $Yv_{8\%}^2\ddot{a}_{\overline{20}|@3.9230485\%}$.

Need to find Y where:

10,000 = 2 * Y * 1.08^-2 * adue(2):<10>

and:

adue(2):<10> = (1 – 1.08^-10) / d(2) = (1 – 1.08^-10) / 0.075499 = 7.11011

d(2) is taken from the Tables.

So: Y = £820.24

## Solution X1.8

*Real and money rates of interest are covered in Chapter 3.*

A real rate of interest is used when inflation needs to be taken into account. [1]

A money rate of interest is used when inflation does not need to be taken into account. [1]

An example for a real rate of interest: A person is saving to go on a round the world trip, which they have calculated will cost £10,000 today. [1]

An example for a money rate of interest: A person has a loan of £10,000, which needs to be paid back in full in 5 years' time. [1]

[Total 4]

*Allow any suitable examples here.*

## Solution X1.9

*Nominal interest and discount are covered in Chapter 2, Section 1.*

(i) ***Rate of discount convertible quarterly***

Using the formula $1+i=\left(1+\dfrac{i^{(p)}}{p}\right)^p$ gives:

$$1+i=\left(1+\frac{0.1}{4}\right)^4=1.10381$$ [1]

*Alternatively* $(1+i)^{¼}=1.025$.

Now using the formula $d^{(p)} = p\left[1-(1+i)^{-\frac{1}{p}}\right]$ gives:

$$d^{(4)} = 4\left[1-(1+i)^{-¼}\right] = 4\left[1-1.025^{-1}\right] = 9.76\% \qquad [1]$$

[Total 2]

*Note: Although* $d = \dfrac{i}{1+i}$*, this formula does not work for nominal rates, ie:* $d^{(4)} \neq \dfrac{i^{(4)}}{1+i^{(4)}}$.
*However it does work for effective rates so it is true that:*

$$\frac{d^{(4)}}{4} = \frac{i^{(4)}/4}{1+i^{(4)}/4}$$

We have:

d(4)/4 = [i(4)/4] / [1 + i(4)/4]

So:

d(4) = 4 * 0.025 / 1.025 = 9.75610% pa

(ii) ***Accumulated amount***

Let $i_1$ and $i_2$ be the effective annual rates for the first and second years respectively. Then using the formula $\dfrac{1}{1+i} = \left(1-\dfrac{d^{(p)}}{p}\right)^p$ gives:

$$1+i_1 = \left(1-\frac{0.06}{2}\right)^{-2} = 1.06281 \qquad [1]$$

*Alternatively* $(1+i_1)^{-\frac{1}{2}} = 0.97$.

Now using the formula $1+i = \left(1+\dfrac{i^{(p)}}{p}\right)^p$ gives:

$$1+i_2 = \left(1+\frac{0.06}{3}\right)^3 = 1.061208 \qquad [1]$$

*Alternatively* $(1+i_2)^{\frac{1}{3}} = 1.02$.

So accumulating the £500:

$$500(1+i_1)(1+i_2) = 500 \times 1.06281 \times 1.061208 = £563.93 \qquad [1]$$

[Total 3]

*Alternatively, we could use the other factors that we calculated:*

$$500(1+i_1)(1+i_2) = 500 \times 0.97^{-2} \times 1.02^3 = £563.93$$

*Note: It is easy to become confused between a quarter (3 months) and the 4 months given in the question. Many students incorrectly use a quarterly time period.*

After 2 years the accumulated amount will be:

500 * (1 – 0.06 / 2)^-2 * (1 + 0.06 / 3)^3 = £563.93

## Solution X1.10

*The interest and discount rates used in this question are covered in Chapter 1, Sections 1 and 3, and in Chapter 2, Sections 1 and 2.*

(i) ***Effective rate***

$$(1+i)^2 = 1.127 \quad \Rightarrow \quad i = 1.127^{\frac{1}{2}} - 1 = 6.16026\%$$

So the equivalent effective annual rate of interest is 6.16%. [1]

[Total 1]

*Don't get confused. This is an **effective** rate paid every two years not a nominal rate convertible every two years as in part (v).*

1.127^0.5 – 1 = 6.16026% pa

(ii) ***Effective rate***

$$v = 1 - d = 0.9425 \quad \Rightarrow \quad (1+i) = 0.9425^{-1} \quad \Rightarrow \quad i = 6.1008\%$$

So the equivalent effective annual rate of interest is 6.10%. [1]

[Total 1]

*Very often it is easier to turn a rate of discount into a rate of interest by first finding* $v$.

*Alternatively, we could use* $i = \dfrac{d}{1-d}$.

(1 – 0.0575)^-1 – 1 = 6.10080% pa

(iii) ***Effective rate***

$i = e^{12\times 0.005} - 1 = 6.18365\%$

So the equivalent effective annual rate of interest is 6.18%. [1]

[Total 1]

| |
|---|
| exp(0.005 * 12) – 1 = 6.18365% pa |

(iv) ***Effective rate***

Using $\dfrac{1}{1+i} = \left(1 - \dfrac{d^{(4)}}{4}\right)^4$, we have:

$$i = \left(1 - \frac{0.06}{4}\right)^{-4} - 1 = 6.23193\%$$

So the equivalent effective annual rate of interest is 6.23%. [1]

[Total 1]

| |
|---|
| (1 – 0.06 / 4)^-4 – 1 = 6.23193% pa |

(v) ***Effective rate***

Using the formula $1 + i = \left(1 + \dfrac{i^{(p)}}{p}\right)^p$, we have:

$$i = \left(1 + \frac{0.14}{\frac{1}{2}}\right)^{\frac{1}{2}} - 1 = 13.1371\%$$

So the equivalent effective annual rate of interest is 13.14%. [1]

[Total 1]

| |
|---|
| (1 + 0.14*2)^0.5 – 1 = 13.13708% pa |

## Solution X1.11

| |
|---|
| *Compound increasing annuities are covered in Chapter 6, Section 2.2.* |

We will work in years, with $v_1 = \dfrac{1}{1.06}$ denoting the one-year discount factor applicable in each of the first 10 years, and $v_2 = \dfrac{1}{1.04}$ denoting the one-year discount factor applicable in each of the final 10 years.

The times, payments and discount factors are:

| Time in years | Time in half-years | Payment | Discount factor |
|---|---|---|---|
| 0.5 | 1 | $250$ | $v_1^{0.5}$ |
| 1 | 2 | $250\times 1.02$ | $v_1$ |
| 1.5 | 3 | $250\times 1.02^2$ | $v_1^{1.5}$ |
| | | | |
| 10 | 20 | $250\times 1.02^{19}$ | $v_1^{10}$ |
| 10.5 | 21 | $250\times 1.02^{20}$ | $v_1^{10}v_2^{0.5}$ |
| 11 | 22 | $250\times 1.02^{21}$ | $v_1^{10}v_2$ |
| | | | |
| 20 | 40 | $250\times 1.02^{39}$ | $v_1^{10}v_2^{10}$ |

The total present value of the payments received is therefore:

$$250v_1^{0.5}+250(1.02)v_1+250(1.02)^2 v_1^{1.5}+\dots+250(1.02)^{19}v_1^{10}$$
$$+250(1.02)^{20}v_1^{10}v_2^{0.5}+250(1.02)^{21}v_1^{10}v_2+\dots+250(1.02)^{39}v_1^{10}v_2^{10} \quad [2]$$

The terms in the first line of the present value expression form a geometric progression of 20 terms, with first term $250v_1^{0.5}$ and common ratio $(1.02)v_1^{0.5}$, so:

$$250v_1^{0.5}+250(1.02)v_1+250(1.02)^2 v_1^{1.5}+\dots+250(1.02)^{19}v_1^{10}$$
$$=250v_1^{0.5}\frac{\left(1-\left(1.02v_1^{0.5}\right)^{20}\right)}{1-1.02v_1^{0.5}}$$
$$=4,450.86 \quad [2]$$

The terms in the second line of the present value expression form a geometric progression of 20 terms, with first term $250(1.02)^{20}v_1^{10}v_2^{0.5}$ and common ratio $(1.02)v_2^{0.5}$, so:

$$250(1.02)^{20}v_1^{10}v_2^{0.5}+250(1.02)^{21}v_1^{10}v_2+\dots+250(1.02)^{39}v_1^{10}v_2^{10}$$
$$=250(1.02)^{20}v_1^{10}v_2^{0.5}\frac{\left(1-\left(1.02v_2^{0.5}\right)^{20}\right)}{1-1.02v_2^{0.5}}$$
$$=4,075.60 \quad [2]$$

So the total present value of the payments is:

$$4,450.86 + 4,075.60 = £8,526.46$$ [1]

[Total 7]

*Alternatively, we could work in half-years, with* $v_3 = \frac{1}{1.06^{0.5}}$ *denoting the half-year discount factor applicable in each of the first 10 years, and* $v_4 = \frac{1}{1.04^{0.5}}$ *denoting the half-year discount factor applicable in each of the final 10 years.*

*Then the total present value of the payments received is:*

$$250v_3 + 250(1.02)v_3^2 + 250(1.02)^2 v_3^3 + \cdots + 250(1.02)^{19} v_3^{20}$$
$$+ 250(1.02)^{20} v_3^{20} v_4 + 250(1.02)^{21} v_3^{20} v_4^2 + \cdots + 250(1.02)^{39} v_3^{20} v_4^{20}$$

*The terms in the first line of the present value expression form a geometric progression of 20 terms, with first term* $250v_3$ *and common ratio* $(1.02)v_3$*, giving a total of:*

$$250v_3 \frac{\left(1-\left(1.02v_3\right)^{20}\right)}{1-1.02v_3} = 4,450.86$$

*The terms in the second line of the present value expression form a geometric progression of 20 terms, with first term* $250(1.02)^{20} v_3^{20} v_4$ *and common ratio* $(1.02)v_4$*, giving a total of:*

$$250(1.02)^{20} v_3^{20} v_4 \frac{\left(1-\left(1.02v_4\right)^{20}\right)}{1-1.02v_4} = 4,075.60$$

*So the total present value of the payments is:*

$$4,450.86 + 4,075.60 = £8,526.46$$

*as before.*

*Alternatively, the sum:*

$$250v_3 + 250(1.02)v_3^2 + 250(1.02)^2 v_3^3 + \cdots + 250(1.02)^{19} v_3^{20}$$

*can be evaluated as an annuity as:*

$$\frac{250}{1.02} a_{\overline{20}|\,@\,j_1} \quad \textit{where} \quad j_1 = \frac{1.06^{0.5}}{1.02} - 1 = 0.93755\%$$

*and the sum:*

$$250(1.02)^{20} v_3^{20} v_4 + 250(1.02)^{21} v_3^{20} v_4^2 + \cdots + 250(1.02)^{39} v_3^{20} v_4^{20}$$

*can be evaluated as an annuity as:*

$$250(1.02)^{19} v_3^{20}\, a_{\overline{20}|\,@\,j_2} \quad \textit{where} \quad j_2 = \frac{1.04^{0.5}}{1.02} - 1 = -0.019225\%$$

PV of the first 10 years' payments is:

PV(0,10) = 250 * v^0.5 + 250 * 1.02 * v + 250 * (1.02^2) * v^1.5 + …{total of 20 terms}

where: v = 1.06^-1.

Using sum of geometric series:

PV(0,10) = 250 * v^0.5 * [1 – (1.02 * v^0.5)^20] / (1 – 1.02 * v^0.5) = 4,450.86

PV at time 0 of the second 10 years' payments is:

PV(10,20) = (1.06^-10) * [250 * (1.02^20) * v^0.5 + 250 * (1.02^21) * v + …{total of 20 terms}]

where: v = 1.04^-1.

Using sum of geometric series:

PV(10,20) = (1.06^-10) * 250 * (1.02^20) * v^0.5 * [1 – (1.02 * v^0.5)^20] / (1 – 1.02 * v^0.5) = 4,075.60

So the total PV is 4,450.86 + 4,075.60 = 8,526.46

## Solution X1.12

*The force of interest as a function of time is covered in Chapter 2, Section 4.*

(i) ***Equivalent constant force of interest***

The accumulation factor from $t=0$ to $t=10$ is:

$$A(0,10)=e^{\int_0^{10} 0.03-0.005t+0.001t^2\,dt}$$

$$=e^{\left[0.03t-0.0025t^2+\frac{0.001}{3}t^3\right]_0^{10}} \quad [1]$$

$$=e^{0.383} \quad [1]$$

Let $\delta$ be the constant force of interest, then:

$$e^{10\delta}=e^{0.383} \quad \Rightarrow \quad 10\delta=0.383 \quad \Rightarrow \quad \delta=3.83\% \quad [1]$$

[Total 3]

Call D the constant force of interest over the 10 years. The accumulation A(0,10) should be the same in both cases so:

exp[D * 10] = exp[INT(0,10):{0.03 – 0.005t + 0.001t^2}dt]

Take logs of both sides:

D*10 = [0.03t – 0.0025t^2 + (0.001t^3) / 3]:(0,10) = 0.3 – 0.25 + 1/3 – 0 = 0.38333

So: D = 0.038333

(ii) ***Accumulated value***

The accumulated value is:

$$250e^{\int_0^{7} 0.03-0.005t+0.001t^2\,dt}+150e^{\int_5^{7} 0.03-0.005t+0.001t^2\,dt} \quad [2]$$

$$=250e^{\left[0.03t-0.0025t^2+\frac{0.001}{3}t^3\right]_0^{7}}+150e^{\left[0.03t-0.0025t^2+\frac{0.001}{3}t^3\right]_5^{7}}$$

$$=250e^{0.20183}+150e^{0.20183-0.12917} \quad [1]$$

$$=250e^{0.20183}+150e^{0.07267}$$

$$=305.91+161.31 \quad [1]$$

$$=£467.22 \quad [1]$$

[Total 5]

*Alternatively, we could accumulate 250 to time 5, then add 150, then accumulate further to time 7.*

This is:

AV = 250 * exp[INT(0,7)] + 150 * EXP[INT(5,7)]

where:

INT(0,7) = [0.03t – 0.0025t^2 + (0.001t^3) / 3]:(0,7) = 0.21 – 0.1225 + 0.11433 – 0 = 0.20183

INT(5,7) = [0.03t – 0.0025t^2 + (0.001t^3) / 3]:(5,7)

= 0.21 – 0.1225 + 0.11433 – (0.15 – 0.0625 + 1/24) = 0.072663

So: AV = 467.22

## Solution X1.13

*Simple increasing annuities are covered in Chapter 6, Section 1.*

As the payments decrease each month, we work in months with a monthly effective interest rate of:

$$1.06^{1/12} - 1 = 0.486755\% \qquad [½]$$

A timeline showing the payments is as follows:

| | | | | | |
|---|---|---|---|---|---|
| | 300 | 290 | ... | 70 | Cashflow |
| 0 | 1 | 2 | ... | 24 | Time (months) |

The present value of the payments (working in months) is therefore:

$$310a_{\overline{24|}} - 10(Ia)_{\overline{24|}} \qquad [1]$$

*Alternatively, the present value of the payments may be written as:*

$$300a_{\overline{24|}} - 10v(Ia)_{\overline{23|}}$$

Now:

$$a_{\overline{24|}} = \frac{1-1.00486755^{-24}}{0.00486755} = 22.599367 \qquad [1]$$

and:

$$(Ia)_{\overline{24|}} = \frac{\left(\dfrac{1-1.00486755^{-24}}{0.00486755/1.00486755}\right) - 24 \times 1.00486755^{-24}}{0.00486755} = 277.235061 \qquad [1]$$

So the present value is:

$$310 \times 22.599367 - 10 \times 277.235061 = £4,233.63 \qquad [½]$$

[Total 4]

Work in months, using a monthly effective interest rate of:

im = 1.06^(1/12) – 1 = 0.48676%

The present value is:

PV = 310 * a:<24> – 10 * (Ia):<24>

where:

a:<24> = (1 – (1+im)^-24) / im = 22.59937

adue:<24> = (1+im) * a:<24> = 22.70937

(Ia):<24> = [22.70937 – (24 * (1+im)^-24)] / im = 277.23506

So: PV = 4,233.45

## Solution X1.14

*Simple increasing annuities are covered in Chapter 6, Section 1. Compound increasing annuities are covered in Chapter 6, Section 2.2.*

For Option 1, a timeline showing the payments made is as follows:


The present value of the annuity under Option 1 is therefore:

$$48\bar{a}_{\overline{15|}} + 2(I\bar{a})_{\overline{15|}} \qquad [1]$$

*Alternatively, this can be expressed as:*

$$50\bar{a}_{\overline{15}|} + 2v(I\bar{a})_{\overline{14}|}$$

Now:

$$\bar{a}_{\overline{15}|} = \frac{1-v^{15}}{\delta} = \frac{1-1.07^{-15}}{\ln(1.07)} = 9.42310 \qquad [1]$$

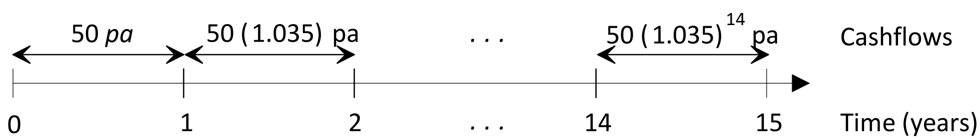
and:

$$(I\bar{a})_{\overline{15}|} = \frac{\left(\dfrac{1-1.07^{-15}}{0.07/1.07}\right) - 15\times 1.07^{-15}}{\ln(1.07)} = 63.68406 \qquad [1]$$

So the present value of the annuity under Option 1 is:

$$48\times 9.42310 + 2\times 63.68406 = £579.68 \qquad [½]$$

For Option 2, a timeline showing the payments made is as follows:




The present value of the annuity under Option 2 is therefore:

$$50\bar{a}_{\overline{1}|} + 50\times 1.035v\,\bar{a}_{\overline{1}|} + \cdots + 50\times 1.035^{14}v^{14}\bar{a}_{\overline{1}|} = 50\bar{a}_{\overline{1}|}\left(1+1.035\,v+\cdots+1.035^{14}v^{14}\right) \qquad [2]$$

The terms in brackets form a geometric progression of 15 terms with first term 1 and common ratio $1.035v$ , so:

$$1+1.035\,v+\cdots+1.035^{14}v^{14} = \frac{1-(1.035v)^{15}}{1-1.035v} = \frac{1-(1.035\times 1.07^{-1})^{15}}{1-1.035\times 1.07^{-1}} = 12.007738 \qquad [1]$$

*Alternatively, this summation can be calculated as:*

$$\ddot{a}_{\overline{15}|\,@\,j} \quad \textit{where} \quad j = \frac{1.07}{1.035} - 1 = 3.381643\%$$

Also:

$$\bar{a}_{\overline{1}|} = \frac{1-v}{\delta} = \frac{1-1.07^{-1}}{\ln(1.07)} = 0.96692 \qquad [½]$$

So the present value of the annuity under Option 2 is:

$$50\times 0.96692\times 12.007738=£580.53$$ [½]

Since the present value of the payments is higher under Option 2, it provides the better deal for the investor. [½]

[Total 8]

The better deal is the option that has the higher present value (PV).

We define the force of interest to be:

D = ln(1.07)

The PV of Option 1 is:

PV(1) = 2 * (Iabar):<15> + 48 * abar:<15>

where:

abar:<15> = (1 – 1.07^-15) / D = 9.42310

For (Iabar) we will need:

adue:<15> = (1 – 1.07^-15) * 1.07/0.07 = 9.74547

So:

(Iabar):<15> = (9.74547 – 15 * 1.07^-15) / D = 63.68406

So: PV(1) = 579.68

The PV of Option 2 is:

PV(2) = 50 * abar:<1> + 50 * abar:<1> * 1.035 * v + … {total of 15 terms}

where:

1.035 * v = 1.035/1.07 = 0.96729

abar:<1> = (1 – 1.07^-1) / d = 0.96692

Using the sum of a geometric series:

PV(2) = 50 * abar:<1> * [1 – (1.035 * v)^15] / (1 – 1.035 * v) = 580.53

So Option 2 provides the better deal.

## Solution X1.15

*The force of interest as a function of time is covered in Chapter 2, Section 4. Payment streams are covered in Chapter 4, Sections 1 and 2.*

Let $A(t_1,t_2)$ denote the accumulated value at time $t_2$ of 1 unit paid at time $t_1$.

(i) ***Accumulated amount***

For $0 \le t < 6$:

$$\int_0^t \delta(s)\,ds = \int_0^t (0.04+0.005s)\,ds = [0.04s+0.0025s^2]_0^t = 0.04t+0.0025t^2 \qquad [1]$$

So:

$$A(0,t) = e^{0.04t+0.0025t^2} \quad (0 \le t < 6) \qquad [1]$$

and:

$$A(0,6) = e^{0.33}$$

For $6 \le t < 8$:

$$\int_6^t \delta(s)\,ds = \int_6^t (0.16-0.015s)\,ds$$

$$= [0.16s-0.0075s^2]_6^t$$

$$= 0.16(t-6)-0.0075(t^2-6^2)$$

$$= -0.69+0.16t-0.0075t^2 \qquad [1]$$

So:

$$A(0,t) = e^{0.33}e^{-0.69+0.16t-0.0075t^2} = e^{-0.36+0.16t-0.0075t^2} \quad (6 \le t < 8) \qquad [1]$$

and:

$$A(0,8) = e^{0.44}$$

For $8 \le t$:

$$\int_8^t \delta(s)\,ds = \int_8^t 0.04\,ds = 0.04(t-8) \qquad [1]$$

So:

$$A(0,t) = e^{0.44}e^{0.04(t-8)} = e^{0.12+0.04t} \quad (8 \le t) \qquad [1]$$

In summary:

$$A(0,t)=\begin{cases} e^{0.04t+0.0025t^2} & 0\le t<6 \\ e^{-0.36+0.16t-0.0075t^2} & 6\le t<8 \\ e^{0.12+0.04t} & 8\le t \end{cases}$$

[Total 6]

For 0 <= t <6:

A(0,t) = exp[INT(0,t):{0.04 + 0.005s}ds]

= exp[0.04s + 0.0025s^2]:(0,t)

= exp[0.04t + 0.0025t^2]

For 6 <= t < 8:

A(0,t) = exp[0.04*6 + 0.0025*6^2] * exp[INT(6,t):{0.16 – 0.015s}ds]

= exp(0.33) * exp[0.16s – 0.0075s^2]:(6,t)

= exp(0.33) * exp[0.16t – 0.0075t^2 – 0.69]

= exp[0.16t – 0.0075t^2 – 0.36]

For t >= 8:

A(0,t) = exp[0.16*8 – 0.0075*8^2 – 0.36] * exp[0.04*(t-8)]

= exp[0.12 + 0.04t]

(ii)(a) ***Present value***

The present value is:

$$\frac{100}{A(0,9)}=\frac{100}{e^{0.12+0.04\times 9}}=\frac{100}{e^{0.48}}=100e^{-0.48}=£61.88 \qquad [1½]$$

*Alternatively, we could calculate this from first principles:*

$$PV=100e^{-\int_0^6 0.04+0.005t\,dt}\,e^{-\int_6^8 0.16-0.015t\,dt}\,e^{-\int_8^9 0.04\,dt}$$

$$=100e^{-\left[0.04t+0.0025t^2\right]_0^6}e^{-\left[0.16t-0.0075t^2\right]_6^8}e^{-\left[0.04t\right]_8^9}$$

$$=100e^{-0.33}e^{-(0.8-0.69)}e^{-(0.36-0.32)}$$

$$=100e^{-0.48}=£61.88$$

This is:

100 / A(0,9) = 100 * exp(– 0.12 – 0.04 * 9) = £61.88

(ii)(b) ***Equivalent annual effective rate of discount***

Letting $d$ denote the equivalent annual effective rate of discount:

$$100e^{-0.48} = 100(1-d)^9 \quad \Rightarrow \quad e^{-0.48} = (1-d)^9 \qquad [1]$$

Therefore:

$$d = 1 - \left(e^{-0.48}\right)^{1/9} = 0.051936 \quad ie \quad 5.1936\% \qquad [½]$$

[Total 3]

*Alternatively, we could calculate the equivalent annual effective interest rate first:*

$$100e^{-0.48} = 100v^9 \quad \Rightarrow \quad i = \left(e^{0.48}\right)^{1/9} - 1 = 0.054781$$

*We can then use this to calculate the annual effective rate of discount:*

$$d = \frac{i}{1+i} = \frac{0.054781}{1.054781} = 0.051936 \quad ie \quad 5.1936\%$$

Need to find d such that:

100 * (1 – d)^9 = 61.88

So:

d = 1 – 0.6188^(1/9) = 5.19361% pa

(iii) ***Present value of payment stream at time 4***

The present value at time 10 is:

$$PV_{t=10} = \int_{10}^{15} 45e^{0.01t} e^{-\int_{10}^{t} 0.04 ds} \, dt \qquad [2]$$

Carrying out the integration:

$$PV_{t=10} = \int_{10}^{15} 45e^{0.01t} e^{-[0.04s]_{10}^{t}} \, dt = \int_{10}^{15} 45e^{0.01t} e^{-0.04t+0.4} \, dt = 45e^{0.4} \int_{10}^{15} e^{-0.03t} \, dt$$

$$= 45e^{0.4} \left[ \frac{e^{-0.03t}}{-0.03} \right]_{10}^{15} = \frac{45e^{0.4}}{-0.03} \left( e^{-0.03\times 15} - e^{-0.03\times 10} \right) = 230.912 \qquad [2]$$

We now need to discount this back to time 4.

The present value at time 4 is:

$$PV_{t=4} = \frac{230.912}{A(4,10)}$$

where:

$$A(4,10) = \frac{A(0,10)}{A(0,4)} = \frac{e^{0.12+0.04\times 10}}{e^{0.04\times 4+0.0025\times 4^2}} = \frac{e^{0.52}}{e^{0.2}} = e^{0.32} \qquad [1]$$

So the present value of this payment stream at time 4 is:

$$PV_{t=4} = \frac{230.912}{A(4,10)} = 230.912e^{-0.32} = 167.677 \qquad [1]$$

[Total 6]

*Alternatively, the discount factor from time 10 to time 4 could be calculated from first principles:*

$$e^{-\int_4^6 0.04+0.005t\,dt} \; e^{-\int_6^8 0.16-0.015t\,dt} \; e^{-\int_8^{10} 0.04\,dt}$$

$$= e^{-\left[0.04t+0.0025t^2\right]_4^6} e^{-\left[0.16t-0.0075t^2\right]_6^8} e^{-\left[0.04t\right]_8^{10}}$$

$$= e^{-(0.33-0.2)} e^{-(0.8-0.69)} e^{-(0.4-0.32)}$$

$$= e^{-0.32}$$

*Alternatively, the answer could be calculated by first calculating the present value at time 0, using the integral:*

$$PV_{t=0} = \int_{10}^{15} 45e^{0.01t} \frac{1}{A(0,t)}\,dt = \int_{10}^{15} 45e^{0.01t}\, e^{-0.12-0.04t}\,dt = 137.282$$

*and then accumulating this forward to time 4:*

$$PV_{t=4} = PV_{t=0} \times A(0,4) = 137.282e^{0.04\times 4+0.0025\times 4^2} = 137.282e^{0.2} = 167.677$$

*where the expressions for* $A(0,t)$ *are taken from part (i).*

First calculate the PV of the payment stream at time 0:

PV(0) = INT(10,15):{45 * exp(0.01*t) * exp[–0.12 – 0.04 * t]}dt

= 45 * exp(–0.12) * INT(10,15):exp(–0.03 * t)dt

= –(45/0.03) * exp(–0.12) * [exp(–0.03 * 15) – exp(–0.03 * 10)]

= 137.28

Then accumulate this to time 4 by multiplying by A(0,4):

PV(4) = 137.28 * exp[0.04*4 + 0.0025 * (4^2)] = 167.68